

Quantum Cryptography**Short notes, fall 2012**

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Important note: These notes are not supposed to be self-contained. Instead, they are intended as a reminder about which topics were discussed in the lecture. If you find mistakes in these notes, please send them to unruh@ut.ee.

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1 Linear Algebra

In the following, we refresh the basic definitions from linear algebra that will be needed during the course. In all definitions, we will restrict our attention to the finite dimensional case only.

Four ways of computing AB :

- (Rows of A) \times B , e.g.,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+0 \\ 3+4 & 6+0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 6 \end{bmatrix}.$$

- $A \times$ (columns of B), e.g.,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+0 \\ 3+4 & 6+0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 6 \end{bmatrix}.$$

- (Rows of A) \times (columns of B), i.e., dot product, e.g.,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+0 \\ 3+4 & 6+0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 6 \end{bmatrix}.$$

- (Columns of A) \times (rows of B), e.g.,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 6 \end{bmatrix}.$$

Gaussian Elimination Gaussian elimination is an algorithm that can be used to solve systems of linear equations, to find the rank of a matrix, to calculate the determinant of a matrix, and to compute the inverse of an invertible square matrix. Typically, the process of Gaussian elimination uses elementary row operations to reduce a matrix to row echelon form, denoting as:

$$EA = U,$$

where E is elimination matrix, which records the row operations, and U is an upper triangular matrix.

Decomposition A matrix can be written as the product of a lower triangular matrix and an upper triangular matrix, i.e.,

$$A = LU$$

Definition 1 (Rank) The column rank of a matrix A is the maximum number of linearly independent column vectors of A . The row rank of a matrix A is the maximum number of linearly independent row vectors of A .¹

Definition 2 (Matrix Inverse) The inverse of a square matrix A is a matrix A^{-1} such that

$$A^{-1}A = I,$$

where I is the identity matrix.

One way to compute A^{-1} :

$$E[A|I] = [I|A^{-1}]$$

Transpose The transpose of a matrix A is denoted as A^T such that

$$(A^T)_{ij} = A_{ji}.$$

Permutation Matrix A permutation matrix, denoted as P , is a square binary matrix that has exactly one entry 1 in each row and each column and 0's elsewhere. A permutation matrix has the following properties:

$$P^{-1} = P^T \quad P^T P = I.$$

Definition 3 (Hilbert space) The n -dimensional Hilbert space is \mathbb{C}^n , the n -dimensional complex vector space.²

\mathbb{C}^n is endowed with the following inner product:

$$\langle \Psi, \Phi \rangle := \sum_{i=1}^n \Psi_i^* \Phi_i$$

where x^* is the complex conjugate of x .³

The (Euclidean) norm $\|\cdot\|$ is defined by

$$\|\Psi\| := \sqrt{\langle \Psi, \Psi \rangle} = \sqrt{\sum_{i=1}^n \Psi_i^* \Psi_i} = \sqrt{\sum_{i=1}^n |\Psi_i|^2}.$$

We call two vectors Ψ and Φ orthogonal if $\langle \Psi, \Phi \rangle = 0$. We call Ψ orthogonal to a subspace $V \subseteq \mathbb{C}^n$ if Ψ is orthogonal to all $x \in V$.

¹Equivalently, the column rank of A is the dimension of the column space of A , while the row rank of A is the dimension of the row space of A .

²Or any complex vector space isomorphic to \mathbb{C}^n

³I.e., $(a + bi)^* = a - bi$.

Furthermore, we call a vector *normalised* if $\|\Psi\| = 1$, and we call a *set* of vectors *orthogonal* if they are pairwise orthogonal, and we call a set of vectors *orthonormal* if they are all normalised and pairwise orthogonal.

Definition 4 (Conjugate transpose) Given a matrix $M \in \mathbb{C}^{n \times m}$, we define M^\dagger as the complex conjugate of the transposition of M , i.e., $(M^\dagger)_{ij} = (M_{ji})^*$. (This is the analogue of transposition.)

We have $(M^\dagger)^\dagger = M$ and $\langle Mx, y \rangle = \langle x, M^\dagger y \rangle$ (and vice-versa).

Definition 5 (Dirac notation) In the Dirac notation, a vector Ψ in \mathbb{C}^n is written $|\Psi\rangle$. By $\langle\Psi|$ we denote the function mapping $|\Phi\rangle$ to $\langle\Psi, \Phi\rangle$ (or equivalently: $\langle\Psi|$ is the row vector $|\Psi\rangle^\dagger$).

In particular, we can now write $\langle\Psi|\Phi\rangle$ for the inner product $\langle\Psi, \Phi\rangle$. And for the projection P_V onto $V = \text{span } \Psi$ we write $P_V = |\Psi\rangle\langle\Psi|$. (Try it out and evaluate $P_V|\Phi\rangle$!)

Definition 6 (Trace) The trace $\text{tr } M$ of a matrix $M \in \mathbb{C}^{n \times n}$ is $\sum_i M_{ii}$.

The trace can also be computed as $\sum_i \langle i|M|i\rangle$ for any orthonormal basis $|1\rangle, \dots, |n\rangle$ of \mathbb{C}^n .

Definition 7 (Hermitian matrices) A matrix $M \in \mathbb{C}^{n \times n}$ is called *Hermitian*, if $M = M^\dagger$. (This is the analogue of symmetric matrices.)

A Hermitian matrix M can be diagonalised, i.e., there is an orthonormal basis $|1\rangle, \dots, |n\rangle$ such that $M = \sum_i \lambda_i |i\rangle\langle i|$ where λ_i are the eigenvalues of M .

Definition 8 (Positive matrices) A matrix $M \in \mathbb{C}^{n \times n}$ is *positive* if for all $|\Psi\rangle \in \mathbb{C}^n$ we have $\langle\Psi|M|\Psi\rangle \geq 0$.

Note that positive is meant in the sense of positive semidefinite (or nonnegative), i.e., we allow, e.g., $M = 0$.

A positive Hermitian matrix has only nonnegative eigenvalues $\lambda_i \geq 0$.

Definition 9 (Absolute value of a matrix) For a positive Hermitian matrix M , let \sqrt{M} be the positive matrix satisfying $(\sqrt{M})^\dagger(\sqrt{M}) = M$. For a (not necessarily positive or Hermitian) matrix M , we define $|M| := \sqrt{M^\dagger M}$.

The matrix $|M|$ is always positive Hermitian. For a positive Hermitian matrix M , we have $|M| = M$. For a diagonal matrix M , we get $|M|$ by taking the absolute value of every element on the diagonal.

For a positive Hermitian M , we can compute \sqrt{M} by first diagonalising M as UDU^\dagger (with unitary U and diagonal D), and then computing \sqrt{D} (by taking the square root of each diagonal element individually) and then computing $\sqrt{M} = U\sqrt{D}U^\dagger$. Since for a matrix M , we have that $M^\dagger M$ is positive Hermitian, we can use this procedure to compute $|M|$.

Definition 10 (Unitary matrices) A matrix $M \in \mathbb{C}^{m \times n}$ is unitary if $M^\dagger M = MM^\dagger = I$ where I is the identity matrix. (Unitary matrices are the analogue to rotation matrices.)

Note: If M is unitary, then $\|Mx\| = \|x\|$ and $\langle Mx, My \rangle = \langle x, y \rangle$.

Definition 11 (Projections) A matrix $M \in \mathbb{C}^{n \times n}$ is a projection if for all x we have $MMx = Mx$ (or equivalently, $MM = M$).

The orthogonal projection P_V onto a subspace $V \subseteq \mathbb{C}^n$ is defined by $P_V(u + v) = v$ where $v \in V$ and u is orthogonal to V . (Note that any state $x \in \mathbb{C}^n$ can be represented uniquely as such a sum $x = u + v$.)

For a one-dimensional subspace $V = \text{span}\{v\}$ with $\|v\| = 1$, we have that $P_V x = v\langle v, x \rangle$.

Lemma 1 (Singular value decomposition) For any square matrix $A \in \mathbb{C}^{n \times n}$, there are unitary matrices $U, V \in \mathbb{C}^{n \times n}$ and a diagonal matrix $D \in \mathbb{C}^{n \times n}$ with only nonnegative real entries such that $A = UDV$.

Definition 12 (Tensor product) Given two Hilbert spaces $\mathbb{C}^n, \mathbb{C}^m$ with orthonormal bases $B_1 = \{|i\rangle\}, B_2 = \{|j\rangle\}$, the tensor product (or Kronecker product) $\mathbb{C}^n \otimes \mathbb{C}^m$ is the Hilbert space \mathbb{C}^{nm} with basis $B_1 \times B_2 = \{|i, j\rangle\}$.⁴

Given two vectors $|\Psi_1\rangle = \sum_i \alpha_i |i\rangle \in \mathbb{C}^n$ and $|\Psi_2\rangle = \sum_j \beta_j |j\rangle \in \mathbb{C}^m$, their tensor product is given by

$$|\Psi_1\rangle \otimes |\Psi_2\rangle = \sum_{i,j} \alpha_i \beta_j |i, j\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m.$$

Given two linear operations $M_1 : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and $M_2 : \mathbb{C}^m \rightarrow \mathbb{C}^m$, we define the linear operation $M_1 \otimes M_2$ to be the unique linear operation satisfying

$$(M_1 \otimes M_2)|i, j\rangle = (M_1|i\rangle) \otimes (M_2|j\rangle).$$

Further reading: [NC00, Section 2.1]

2 One Qubit

Definition 13 (Qubit) A single qubit is represented by a vector $|\Psi\rangle \in \mathbb{C}^2$ with $\| |\Psi\rangle \| = 1$.

There are two kinds of operations on qubits, unitary transformations and measurements.

⁴There exists a more general category theoretical definition using a universal property, but for our purposes this specialisation is sufficient.

Definition 14 (Unitary transformations on qubit) A unitary transformation on a qubit $|\Psi\rangle$ is represented by a unitary matrix $U \in \mathbb{C}^{2 \times 2}$. The qubit after the transformation is $U|\Psi\rangle$.

In the case of polarisation, a typical transformation would be to rotate the polarisation by an angle of α . In this case we have

$$U = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$

which can be easily verified to be unitary.

Definition 15 (Projective measurements) A projective measurement on a qubit is defined by two orthonormal vectors $|yes\rangle$ and $|no\rangle$. The outcomes of the measurement can be yes or no.⁵

When applying the measurement to a qubit $|\Psi\rangle$, the probability for the yes outcome is $|\langle yes|\Psi\rangle|^2$, and the probability for the no outcome is $|\langle no|\Psi\rangle|^2$.

In case of a yes outcome, the resulting state is $|yes\rangle$, and in case of a no outcome, the resulting state is $|no\rangle$.⁶

An example for a measurement is a polarising filter. If the filter lets only vertically polarised light through, it corresponds to a measurement with $|yes\rangle = |\uparrow\rangle$ and $|no\rangle = |\leftrightarrow\rangle$, and a yes-outcome corresponds to the fact that the photon passes the filter. In this case, the resulting photon will be vertically polarised (i.e., in the $|\uparrow\rangle$ state). (In the no-outcome, the photon is destroyed, so talking about the resulting photon does not make sense in that case.)

A few typical unitary transformations on qubits are:

Definition 16 (Hadamard) The Hadamard gate (usually denoted H) is defined by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

or equivalently

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \\ H|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

The Hadamard gate is useful for introducing superpositions as it takes a classical bit ($|0\rangle$ or $|1\rangle$) and transforms it into a superposition).

⁵Of course, the labelling yes/no is arbitrary. Any other two labels are possible.

⁶Up to a scalar factor of absolute value 1. To be completely exact, the state after measuring yes is $\frac{\langle yes|\Psi\rangle}{|\langle yes|\Psi\rangle|} \cdot |yes\rangle$ (and analogous for no), but this should not worry us now. Furthermore, scalar factors (called a *global phase*) do not have physical meaning anyway.

Definition 17 (Bit flip) *The bit flip (also called not-gate or X-gate) is defined by*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

or equivalently

$$\begin{aligned} X|0\rangle &= |1\rangle \\ X|1\rangle &= |0\rangle \end{aligned}$$

The bit flip corresponds to a negation. It can, however, be applied in superposition.

Definition 18 (Rotation) *The rotation by angle θ is defined by*

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

or equivalently

$$\begin{aligned} R_\theta|0\rangle &= \cos \theta|0\rangle + \sin \theta|1\rangle \\ R_\theta|1\rangle &= -\sin \theta|0\rangle + \cos \theta|1\rangle \end{aligned}$$

Definition 19 (Phase shift) *The phase shift S is defined by*

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

or equivalently

$$\begin{aligned} S|0\rangle &= |0\rangle \\ S|1\rangle &= i|1\rangle \end{aligned}$$

More generally, we can parametrise the phase shift by an angle θ :

$$S_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

or equivalently

$$\begin{aligned} S_\theta|0\rangle &= |0\rangle \\ S_\theta|1\rangle &= e^{i\theta}|1\rangle \end{aligned}$$

Note that $S = S_{\frac{\pi}{2}}$.

Further reading: [NC00, Section 1.2.1, 1.3.1], and [NC00, Section 4.2] for the single qubit gates.

3 Elitzur-Vaidman Bomb Testing

A *beam splitter* is a device into which a photon can enter in two positions (call them *up* and *down*), and exit in two positions (call them *up* and *down*, too). The input to the beam splitter is a qubit that is represented as a superposition between $|\text{up}\rangle$ and $|\text{down}\rangle$. Then the beam splitter performs the following linear transformation $B_{\frac{\pi}{4}}$:

$$\begin{aligned} B_{\frac{\pi}{4}}|\text{up}\rangle &= \frac{1}{\sqrt{2}}(|\text{up}\rangle + |\text{down}\rangle) \\ B_{\frac{\pi}{4}}|\text{down}\rangle &= \frac{1}{\sqrt{2}}(-|\text{up}\rangle + |\text{down}\rangle) \end{aligned}$$

Another variant of the beam splitter is given by the linear transformation

$$\begin{aligned} B_{-\frac{\pi}{4}}|\text{up}\rangle &= \frac{1}{\sqrt{2}}(|\text{up}\rangle - |\text{down}\rangle) \\ B_{-\frac{\pi}{4}}|\text{down}\rangle &= \frac{1}{\sqrt{2}}(|\text{up}\rangle + |\text{down}\rangle) \end{aligned}$$

Note that $B_{\frac{\pi}{4}}$ and $B_{-\frac{\pi}{4}}$ are unitary, and that $B_{\frac{\pi}{4}}B_{-\frac{\pi}{4}} = B_{-\frac{\pi}{4}}B_{\frac{\pi}{4}} = 1$.

The *Elitzur-Vaidman bomb tester* is the following construction. We are given a box that may or may not contain a bomb. The bomb explodes if a single photon falls onto it. We want to find out whether the box contains a bomb. To do so, we take a $B_{\frac{\pi}{4}}$ beam splitter and send an $|\text{up}\rangle$ photon through it. The state that comes out of the beam splitter is $\frac{1}{\sqrt{2}}(|\text{up}\rangle + |\text{down}\rangle)$. Now we put the box in the path of the $|\text{down}\rangle$ photon. Assume for the moment that a bomb is in that box. Then the box constitutes a measurement whether the photon takes the up- or the down-path. Since the state of the photon is $\frac{1}{\sqrt{2}}(|\text{up}\rangle + |\text{down}\rangle)$, the measurement outcome will be up or down, each with probability $\frac{1}{2}$. In the case of a down-outcome, the bomb explodes. In the case of an up-outcome, the resulting state is $|\text{up}\rangle$ (i.e., the photon takes the upper path). Then the photon passes the $B_{-\frac{\pi}{4}}$ beam splitter and is transformed into $\frac{1}{\sqrt{2}}(|\text{up}\rangle - |\text{down}\rangle)$. Now we measure whether the photon is in the up state or the down state (by simply putting a photon detector in at the end of both paths). With probability $\frac{1}{2}$ the photon will be up (conditioned on the fact that the bomb did not explode), with probability $\frac{1}{2}$ it will be down. Altogether we get the following predictions for this experiment.

Event	Probability
Bomb explodes	$\frac{1}{2}$
Photon is in up-path	$\frac{1}{4}$
Photon is in down-path	$\frac{1}{4}$

On the other hand, if no bomb is in the box, the box has no effect on the photon. In this case, the experiment consists of two beam splitters $B_{\frac{\pi}{4}}$ and $B_{-\frac{\pi}{4}}$ in a row. Because these beam splitters are inverses of each other, they cancel each other out, and the photon coming out of the second beam splitter will be in state $|\text{up}\rangle$. Thus in this case we get the following probabilities:

Event	Probability
Bomb explodes	0
Photon is in up-path	1
Photon is in down-path	0

In other words, if the outcome of the experiment is “down”, we know for sure that there is a bomb in the box without having caused it to explode. Unfortunately, with probability $\frac{1}{2}$ the bomb still explodes. The experiment can, however, be improved to make the probability of the bomb exploding arbitrarily small (homework).

Further reading: For the modelling of the beam splitter: [NC00, Section 7.4] (uses some physics we have not discussed yet). For the bomb tester: [Wik, Elitzur-Vaidman bomb-tester].

4 Larger quantum systems

Definition 20 (Quantum states) *An n -dimensional quantum state is represented by a vector $|\Psi\rangle \in \mathbb{C}^n$ with $\| |\Psi\rangle \| = 1$ (here \mathbb{C}^n is a Hilbert space).*

In most cases, we assume some canonical orthonormal basis of \mathbb{C}^n (representing the classical possibilities of the system) which we call the *computational basis*. We then use the following convention: If $|b_1\rangle, \dots, |b_n\rangle$ are the basis vectors, and b_1, \dots, b_n are some labels we assign to these vectors sorted according to some natural ordering (e.g., for an m -qubit system (i.e., $n = 2^m$) b_i is the bitstring $b_i \in \{0, 1\}^m$ which is the binary representation of $i - 1$), then $|b_i\rangle = (0, \dots, 0, 1, 0, \dots, 0)^t$ where the 1 is at the i -th position.

We abbreviate $x \otimes \dots \otimes x$ (n components) as $x^{\otimes n}$. See Definition 12 for the definition of \otimes .

There are two kinds of operations on quantum states, unitary transformations and measurements.

Definition 21 (Unitary transformation) *A unitary transformation on a quantum state $|\Psi\rangle \in \mathbb{C}^n$ is represented by a unitary matrix $U \in \mathbb{C}^{n \times n}$. The state after the transformation is $U|\Psi\rangle$.*

Definition 22 (Measurement) *A (projective) measurement on a Hilbert space \mathcal{H} is specified by a family $\{P_i\}_{i \in I}$ of orthogonal projections on \mathcal{H} labelled with the possible measurement outcomes $i \in I$. The projections have to be pairwise orthogonal, i.e., $P_i P_j = 0$ for $i \neq j$. And the projections sum to 1, i.e., $\sum_i P_i = 1_{\mathcal{H}}$ where $1_{\mathcal{H}}$ is the identity on \mathcal{H} .*

When measuring a state $|\Psi\rangle \in \mathcal{H}$, the outcome i occurs with probability

$$\|P_i |\Psi\rangle\|^2.$$

If the outcome i occurs, the state after the measurement (post-measurement state) is

$$\frac{P_i|\Psi\rangle}{\|P_i|\Psi\rangle\|}.$$

A special case of a measurement is the complete measurement in which every projection is the projection onto a one-dimensional subspace.

Note that we can also represent a measurement by giving the images V_i of the projectors P_i instead of the projectors themselves. This is equivalent, as the P_i can be recovered from V_i and vice versa.

Definition 23 (Complete measurement) A complete measurement on \mathcal{H} is specified by an orthonormal basis $B = \{|i\rangle\}_{i \in I}$ of \mathcal{H} labelled with the possible measurement outcomes $i \in I$.

When measuring a state $|\Psi\rangle \in \mathcal{H}$, the outcome i occurs with probability

$$|\langle i|\Psi\rangle|^2.$$

and the corresponding post-measurement state is

$$\frac{\langle i|\Psi\rangle}{|\langle i|\Psi\rangle|} \cdot |i\rangle$$

(which is $|i\rangle$ up to a (physically irrelevant) scalar factor $\frac{\langle i|\Psi\rangle}{|\langle i|\Psi\rangle|}$ of absolute value 1, the global phase).

Note that the complete measurement with basis $\{|i\rangle\}_{i \in I}$ has the same effect as the measurement with projectors $\{P_i\}_{i \in I}$ where $P_i := |i\rangle\langle i|$. Thus complete measurements are a special case of measurements as in Definition 22.

Further reading: [NC00], Section 2.2.1, 2.2.2, and 2.2.5 for states, unitary evolution, and projective measurements, respectively. Section 2.2.7 for information in the global phase.

5 Multi-qubit gates

Definition 24 (Controlled NOT) The *CNOT* gate on \mathbb{C}^4 is defined to be the linear operation defined by

$$\begin{aligned} \text{CNOT}|00\rangle &= |00\rangle \\ \text{CNOT}|01\rangle &= |01\rangle \\ \text{CNOT}|10\rangle &= |11\rangle \\ \text{CNOT}|11\rangle &= |10\rangle \end{aligned}$$

or equivalently

$$\text{CNOT}|a, b\rangle = |a, a \oplus b\rangle \quad (a, b \in \{0, 1\})$$

where \oplus denotes XOR.

In circuits, we write CNOT as follows:



The dot represents the controlling qubit, and the \oplus represents the qubit that is conditionally flipped. The dot does not have to be on the qubit above the \oplus . For example,



represents the operation defined by

$$|a, b, c\rangle \mapsto |a \oplus c, b, c\rangle \quad (a, b, c \in \{0, 1\})$$

Definition 25 (SWAP) The SWAP gate on \mathbb{C}^4 is defined to be the linear operation defined by

$$\text{SWAP}|a, b\rangle = |b, a\rangle.$$

The swap gate is represented by



Again, the two \times do not have to be on adjacent lines.

Definition 26 (Toffoli) The Toffoli gate on \mathbb{C}^8 is defined to be the linear operation defined by

$$\text{Toffoli}|a, b, c\rangle = |a, b, (a \cdot b) \oplus c\rangle$$

where \cdot is the multiplication modulo 2, or equivalently, the and-operation.

The Toffoli gate is usually represented as follows:

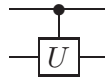


As with the CNOT, the two dots can be on arbitrary lines, not only those adjacent to the \oplus . Furthermore, the symbol generalises to more than two controlling qubits in the obvious way.

Definition 27 (Controlled- U) Given a unitary transformation $U \in \mathbb{C}^n$, the controlled- U gate $C(U)$ is defined to be the linear operation on \mathbb{C}^{2n} defined by

$$\begin{aligned} C(U)|0, j\rangle &= |0, j\rangle \\ C(U)|1, j\rangle &= |1\rangle \otimes U|j\rangle. \end{aligned}$$

The controlled- U is depicted as follows:



Again, the dot can be on an arbitrary qubit.

Further reading: [NC00], Section 4.3

6 Composite Systems

Definition 28 (Composite systems and states) Given n quantum systems \mathcal{H}_i , the composite system is $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$.

Given n quantum states $|\Psi_i\rangle \in \mathcal{H}_i$, the composite state consisting of n independent subsystems in states $|\Psi_i\rangle$ is

$$|\Psi_1\rangle \otimes \dots \otimes |\Psi_n\rangle \in \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n.$$

Definition 29 (Composite unitary operations) Given a composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$, performing the unitary operation U_1 on \mathcal{H}_1 and U_2 on \mathcal{H}_2 independently is equivalent to performing the unitary operation $U_1 \otimes U_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

A special case is performing an operation U only on \mathcal{H}_1 and not touching \mathcal{H}_2 . This is represented by $U \otimes I$ where I is the identity.

Definition 30 (Composite measurements) Given a measurement M_1 specified by projections P_1, \dots, P_n on \mathcal{H}_1 and a measurement M_2 specified by projections P'_1, \dots, P'_m on \mathcal{H}_2 , performing each of the measurements independently is equivalent to performing the measurement M specified by the projections $P_{ij} := P_i \otimes P_j$ with $i = 1, \dots, n$ and $j = 1, \dots, m$. (I.e., the possible outcomes of M are pairs i, j with $i = 1, \dots, n$ and $j = 1, \dots, m$.)

Note that the measurement that does nothing and has no effect on the state is given by the single projector I (the identity). Thus a measurement M on \mathcal{H}_1 only extends to a measurement M' on $\mathcal{H}_1 \otimes \mathcal{H}_2$ as follows: If M consists of P_1, \dots, P_n , then M' consists of $P_1 \otimes I, \dots, P_n \otimes I$.

Further reading: [NC00], Section 2.2.8.

7 Sets of Elementary Gates

The following theorem is a corollary of the so-called Solovay-Kitaev theorem:

Theorem 1 Fix $\varepsilon > 0$. Fix a unitary operation U operating on \mathbb{C}^{2^n} (an n -qubit operation).

Then there exists a $\varphi \in \mathbb{C}$ with $|\varphi| = 1$ (a global phase factor), and a quantum circuit C of size $\text{polylog}(1/\varepsilon) + \text{exp}(n)$ containing only the gates CNOT, H (Hadamard), $R_{\frac{\pi}{8}}$ (rotation by $\frac{\pi}{8}$), S (phase shift) such that the following holds:

For all $|\Psi\rangle \in \mathbb{C}^{2^n}$ with $\|\Psi\| = 1$, we have that

$$\|\varphi U|\Psi\rangle - U_C|\Psi\rangle\| \leq \varepsilon$$

where U_C is the unitary transformation implemented by the circuit C .

In other words, we can approximate any unitary transformation U by a circuit containing only the above-mentioned gates (up to a global phase factor φ which is physically irrelevant). The construction is very efficient in terms of the error ε , but becomes inefficient for larger systems (exponential in the number n of qubits).

As a consequence, we may assume any finite set of elementary gates that is powerful enough to implement CNOT, Hadamard, Rotation by $\frac{\pi}{8}$ and phase shift (up to an arbitrarily small error). The theorem above then implies that it does not matter which set of gates we choose. (Note that in a finite set of gates, the number of qubits a gate operates on is $n = O(1)$, so the exponential term $\exp(n)$ in the complexity of the construction vanishes.)

Fault tolerant computation. In the above, we assumed that we are given error free gates, i.e., the gates always implement the unitary transformation they are supposed to implement. However, in practise we will have very noisy components that introduce errors on the qubits. Fortunately, it is possible to implement quantum circuits in a fault tolerant fashion (under reasonable assumptions about the gates and the error model). Then, given gates that have an error probability of approximately 10^{-5} – 10^{-6} , we can get almost error free computation. In the following, we always assume error free gates and communication for simplicity.

Further reading: [NC00], Appendix 3 for the Solovay-Kitaev theorem (which gives Theorem 1) for the case $n = 2$, and Section 4.5 on how to get a larger value of n .

[NC00], Section 4.6 for fault tolerant computation (needs knowledge of the preceding sections on error correcting codes).

8 The Deutsch-Jozsa Algorithm

Deutsch's algorithm. Assume we are given a function $f : \{0, 1\} \rightarrow \{0, 1\}$. We ask the question which of the following two cases applies:

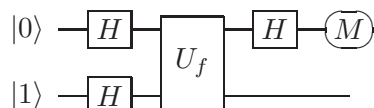
- f is constant ($f(0) = f(1)$), or
- f is balanced ($f(0) \neq f(1)$).

We further assume that f is implemented as a unitary transformation U_f on two qubits that performs the following operation:

$$U_f|x, y\rangle = |x, y \oplus f(x)\rangle \quad (x, y \in \{0, 1\})$$

(Such a unitary can be efficiently implemented if f has a poly-size classical circuit.)

Deutsch's algorithm performs the following operations:



(Here $\text{---}\overline{M}$ denotes a complete measurement of the first qubit in the computational basis, i.e., we look whether it is $|0\rangle$ or $|1\rangle$.)

Computing the output of this circuit, we get the following:

- If f is constant, then with probability 1 the measurement M has outcome 0.
- If f is balanced, then with probability 1 the measurement M has outcome 1.

Thus with one evaluation of f we have determined whether f is constant or balanced. Classically, we would have needed two evaluations.

An extension of this algorithm, the Deutsch-Jozsa algorithm, can even handle functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and decide whether they are constant or balanced (same number of 0 and 1 outputs). It needs only one evaluation of f . (There is no guarantee if f is neither constant nor balanced.)

Further reading: [NC00, Section 1.4.3].

9 Density Operators

Intuitively, a quantum ensemble is a probability distribution on quantum states.

Definition 31 (Quantum ensemble) A quantum ensemble E over a Hilbert space \mathcal{H} is a (possibly infinite) set of pairs $E = \{(|\Psi_i\rangle, p_i)\}_i$ satisfying:

- For all i we have $|\Psi_i\rangle \in \mathcal{H}$.
- The vectors $|\Psi_i\rangle$ are normalized ($\| |\Psi_i\rangle \| = 1$).
- We have $p_i \geq 0$ for all i and $\sum_i p_i = 1$.

The interpretation is that a system is in state $|\Psi_i\rangle$ with probability p_i .

Operations performed on quantum states generalise to ensembles.

Definition 32 (Unitary transformation) Let U be a unitary matrix on \mathcal{H} . Let $E = \{(|\Psi_i\rangle, p_i)\}_i$ be an ensemble over \mathcal{H} .

Then applying U to the ensemble E leads to the ensemble

$$UE = \{(U|\Psi_i\rangle, p_i)\}_i.$$

Definition 33 (Measurement) Let $M = \{Q_1, \dots, Q_n\}$ be a projective measurement over \mathcal{H} consisting of projectors Q_i . Let $E = \{(|\Psi_i\rangle, p_i)\}_i$ be an ensemble over \mathcal{H} .

If we measure the state described by E with M , the outcome j has probability

$$\Pr[\text{Outcome } j] = \sum_i p_i \|Q_j |\Psi_i\rangle\|^2.$$

After measuring the outcome j , the system state is described by the following ensemble:

$$\left\{ \left(\frac{Q_j |\Psi_i\rangle}{\|Q_j |\Psi_i\rangle\|}, \frac{p_i \|Q_j |\Psi_i\rangle\|^2}{\Pr[\text{Outcome } j]} \right) \right\}_i.$$

Definition 34 (Extending the state space) Let $E = \{(|\Psi_i\rangle, p_i)\}_i$ be an ensemble over \mathcal{H} . Let $|\Gamma\rangle \in \mathcal{H}'$, $\| |\Gamma\rangle \| = 1$.

Then extending the state described by E by adding another quantum system described by $|\Gamma\rangle$ results in the following ensemble over $\mathcal{H} \otimes \mathcal{H}'$:

$$E \otimes |\Gamma\rangle = \{(|\Psi_i\rangle \otimes |\Gamma\rangle, p_i)\}_i.$$

Definition 35 (Physical indistinguishability) We call two ensembles physically indistinguishable if all sequences of operations according to Definitions 32, 33, and 34 lead to the same probabilities of measurement outcomes.

A density operator is a compact representation of a quantum ensemble. This representation loses some information contained in the description of an ensemble,⁷ but it still contains enough information to predict the outcome of physical experiments.

Definition 36 (Density operator) Let $E = \{(|\Psi_i\rangle, p_i)\}_i$ be a quantum ensemble over \mathcal{H} . The density operator (density matrix, mixed state) corresponding to E is the linear transformation ρ_E on \mathcal{H} defined as follows:

$$\rho_E = \sum_i p_i |\Psi_i\rangle \langle \Psi_i|.$$

We call ρ a density operator over \mathcal{H} if it is a density operator for some quantum ensemble E over \mathcal{H} . By $S(\mathcal{H})$ we denote the set of all density operators over \mathcal{H} .

Note: The usage of the words *mixed state* and *pure state* is ambiguous. There are two usages:

- A mixed state is a density operator $\rho \in S(\mathcal{H})$ and a pure state is a state described by a vector $|\Psi\rangle \in \mathcal{H}$.
- A pure state is a density operator of the form $|\Psi\rangle \langle \Psi|$ (i.e., a density operator corresponding to an ensemble with only one entry), and a mixed state is a density operator that cannot be written as $|\Psi\rangle \langle \Psi|$.

Lemma 2 The set $S(\mathcal{H})$ consists of all positive Hermitian matrices with trace 1.

Due to its mathematical simplicity, one usually takes Lemma 2 as the definition of density operators.

Definition 37 (Unitary transformation) Let U be a unitary matrix on \mathcal{H} . Let $\rho \in S(\mathcal{H})$ be a density operator over \mathcal{H} .

Then applying U to the state ρ leads to the state $U\rho U^\dagger$.

⁷E.g., the following two ensembles both have the same representation as a density operator: $\{(|0\rangle, \frac{1}{2}), (|1\rangle, \frac{1}{2})\}$ and $\{(|+\rangle, \frac{1}{2}), (|-\rangle, \frac{1}{2})\}$ with $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.

Definition 38 (Measurement) Let $M = \{Q_1, \dots, Q_n\}$ be a projective measurement over \mathcal{H} consisting of projectors Q_i . Let $\rho \in S(\mathcal{H})$ be a density operator over \mathcal{H} .

If we measure the state ρ with M , the outcome j has probability

$$\Pr[\text{Outcome } j] = \text{tr } Q_j \rho Q_j^\dagger = \text{tr } Q_j \rho.$$

After measuring the outcome j , the system state is $\frac{Q_j \rho Q_j^\dagger}{\text{tr } Q_j \rho Q_j^\dagger}$.

Definition 39 (Extending the state space) Let $\rho \in S(\mathcal{H})$ be a density operator over \mathcal{H} .

Then extending the state ρ by adding another quantum system described by $\sigma \in S(\mathcal{H}')$ results in the density operator $\rho \otimes \sigma$ over $\mathcal{H} \otimes \mathcal{H}'$

The following theorem states that density operators characterise physical indistinguishability of quantum ensembles.

Theorem 2 Let E, E' be quantum ensembles over \mathcal{H} and ρ, ρ' the corresponding density operators. Then E and E' are physically indistinguishable if and only if $\rho = \rho'$.

Since in physics, there is no reason to assume that some distinction exists if it is principally impossible to measure it, one usually directly says that the physical system is in the state ρ and does not assume that there is some hidden ensemble behind this state that contains more information than the density operator ρ .

Further reading: [NC00, Section 2.4.1 and 2.4.2]. Note that they define a density operator as being positive Hermitian (and omit the condition $\text{tr } \rho = 1$).

10 Partial Trace and Purification

Definition 40 (Partial trace) Let a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$ be given.

The partial trace $\text{tr}_B : S(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow S(\mathcal{H}_A)$ is the linear transformation defined by

$$\text{tr}_B \sigma \otimes \tau = \sigma \cdot \text{tr } \tau \quad \sigma \in S(\mathcal{H}_A), \tau \in S(\mathcal{H}_B).$$

We say that \mathcal{H}_B (or just B) is traced out. Analogously we can also trace out \mathcal{H}_A or consider multipartite systems.

Given a state $\rho \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$, the state $\rho^A := \text{tr}_B \rho$ describes the state resulting from destroying (or locking away) the B -part of the system. Or equivalently, ρ^A represents all information that can be extracted about the state ρ from the A -part of the system alone.

Theorem 3 (Purification) Let a state $\rho \in S(\mathcal{H}_A)$ be given. Then for any space \mathcal{H}_B such that $\dim \mathcal{H}_B \geq \dim \mathcal{H}_A$, there is a quantum state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$\text{tr}_B |\Psi\rangle\langle\Psi| = \rho.$$

We call Ψ a *purification* of ρ . Note that the purification is not unique.

This theorem means that any mixed state can be considered as a part of some larger pure state (we usually call the added subsystem \mathcal{H}_B the *environment*).

In many cases, analysing a pure system may be simpler than analysing a mixed one. In these cases Theorem 3 allows to simplify the analysis.

Further reading: [NC00, Section 2.4.3] for the partial trace and [NC00, Section 2.5] for purification.

11 Quantum Operations

Definition 41 (Quantum Operations) A quantum operation \mathcal{E} is a map $\mathcal{E} : S(\mathcal{H}) \rightarrow S(\mathcal{H}')$ of the form

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad (1)$$

where $E_k : \mathcal{H} \rightarrow \mathcal{H}'$ are linear operators satisfying $\sum_k E_k^\dagger E_k = I$ (where I is the identity on \mathcal{H}).

We sometimes write $\mathcal{E} = \{E_k\}_k$ to denote the fact that \mathcal{E} is the operation defined by (1). The operator E_k are called the Kraus operators of \mathcal{E} .

Quantum operations describe all operations that can be applied to a mixed state ρ , including unitary transformations, measurements (when the outcomes are erased). Also the partial trace is an example of a quantum operation.

Quantum operations are also called *superoperators*.

Definition 42 (Composing operations) Let \mathcal{E} and \mathcal{F} be two quantum operations (over \mathcal{H}_E and \mathcal{H}_F , respectively). Then $\mathcal{E} \otimes \mathcal{F}$ is the linear operation defined by

$$(\mathcal{E} \otimes \mathcal{F})(\sigma \otimes \tau) = \mathcal{E}(\sigma) \otimes \mathcal{F}(\tau).$$

Note that $\mathcal{E} \otimes \mathcal{F}$ is a quantum operation over $\mathcal{H}_E \otimes \mathcal{H}_F$.

Theorem 4 $\mathcal{E} : S(\mathcal{H}) \rightarrow S(\mathcal{H}')$ is a quantum operation if and only if it satisfies the following three conditions:

- It is linear.
- It is trace-preserving (i.e., $\text{tr } \mathcal{E}(\rho) = \text{tr } \rho$).
- It is completely positive. That is, for any vector space $\tilde{\mathcal{H}}$ and any positive $\rho \in S(\mathcal{H} \otimes \tilde{\mathcal{H}})$, we have that $(\mathcal{E} \otimes I)(\rho)$ is positive, too. (Here I is the identity on $\tilde{\mathcal{H}}$.)

Further reading: [NC00, Section 8.2].

12 Trace distance

Note: In the following, we will use random variables and probability distributions interchangeably. That is, if we say “ X is a probability distribution over A ”, we may then use X as a random variable taking values in A and write $\Pr[X = a]$ for the probability assigned by the distribution X to a .

Definition 43 (Statistical distance) *Let X and Y be probability distributions over some (countable) set A . Then the statistical distance $\text{SD}(X, Y)$ between X and Y is defined as*

$$\text{SD}(X, Y) := \max_{T \subseteq A} |\Pr[X \in T] - \Pr[Y \in T]|.$$

Intuitively, the statistical distance tells us how good a sample chosen according to the distribution X and a sample chosen according to Y can be distinguished by an optimal statistical test T .

Lemma 3 (Alternative definition of statistical distance) *Let X and Y be probability distributions over some (countable) set A . Then*

$$\text{SD}(X, Y) = \frac{1}{2} \sum_{a \in A} |\Pr[X = a] - \Pr[Y = a]|.$$

This lemma is often taken as the definition of statistical distance. However, it does not have an operational meaning like Definition 43 and it does not generalise to uncountable sets A .

The statistical distance is often used in cryptography in definitions of security against computationally unlimited adversaries: If we have some random variable I that describes what the output/communication of the protocol should ideally look like (e.g., it should be stochastically independent of the secrets used in the protocol), and the random variable R describes the actual output/communication, then one would require that $\text{SD}(R, I)$ is sufficiently small.

Lemma 4 • *The statistical distance SD is a metric (on the set of probability distributions over a given set A).*

- *For any (possibly randomized) function F we have that*

$$\text{SD}(F(X), F(Y)) \leq \text{SD}(X, Y)$$

If F is injective, equality holds.

(This means that applying a function to some data may not make it more distinguishable, it may only lose information.)

- *Let X, Y, Z be stochastically independent. Then*

$$\text{SD}((X, Z), (Y, Z)) \leq \text{SD}(X, Y)$$

where (X, Z) is the random variable describing pairs chosen according to X and Z .

(Adding independent information does not help in distinguishing.)

Definition 44 (Trace distance) Given density operators $\sigma, \rho \in S(\mathcal{H})$, we define the trace distance $\text{TD}(\sigma, \rho)$ as

$$\text{TD}(\sigma, \rho) := \frac{1}{2} \text{tr} |\sigma - \rho|.$$

Here $|M|$ denotes the absolute value of the matrix M , see Definition 9.

Lemma 5 (Alternative definition of the trace distance) Given density operators $\sigma, \rho \in S(\mathcal{H})$ we have that

$$\text{TD}(\sigma, \rho) = \max_P |\text{tr} P\sigma - \text{tr} P\rho|.$$

Here P ranges over all orthogonal projectors on \mathcal{H} .

In other words, the trace distance tells us how good we can distinguish the states σ and ρ by a measurement $\{P, 1 - P\}$. This is analogous to Definition 43 since a quantum measurement is the analogue of a statistical test in the classical world.

This analogy is made even stronger by the following lemma:

Lemma 6 Let X and Y be probability distributions over A . Let

$$\rho_X := \sum_{a \in A} \text{Pr}[X = a] |a\rangle\langle a| \in S(\mathbb{C}^A)$$

(in other words, ρ_X describes the distribution X over classical states $|a\rangle$) and ρ_Y analogous.

Then $\text{SD}(X, Y) = \text{TD}(\rho_X, \rho_Y)$.

Lemma 7 • The trace distance TD is a metric (on $S(\mathcal{H})$).

- For any quantum operation \mathcal{E} and any $\sigma, \rho \in S(\mathcal{H})$ we have that

$$\text{TD}(\mathcal{E}(\sigma), \mathcal{E}(\rho)) \leq \text{TD}(\sigma, \rho).$$

If \mathcal{E} applies a unitary (i.e., $\mathcal{E}(\rho) := U\rho U^\dagger$), then equality holds.

- Let $\sigma, \rho \in S(\mathcal{H})$ and $\tau \in S(\mathcal{H}')$. Then

$$\text{TD}(\sigma \otimes \tau, \rho \otimes \tau) = \text{TD}(\sigma, \rho).$$

Note the one-to-one correspondence with the properties in Lemma 4.

Lemma 8 Let P be an orthogonal projector on \mathcal{H} , let $\rho \in S(\mathcal{H})$, let $\varepsilon \geq 0$. Assume that $\text{tr} P\rho \geq 1 - \varepsilon$ (i.e., the measurement $\{P_{\text{yes}} := P, P_{\text{no}} := 1 - P\}$ returns yes with high probability).

Then there is a state $\rho' \in S(\mathcal{H})$ such that

- (a) $\text{TD}(\rho, \rho') \leq \sqrt{\varepsilon}$.

(b) There are states $|\Psi_i\rangle \in \text{im } P$ and values p_i with $\sum_i p_i = 1$, $p_i \geq 0$ such that $\rho' = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$. (In other words, when measuring ρ' , the measurement would always return yes, i.e., ρ' satisfies the property specified by P .)

This lemma gives a criterion to show that the trace distance between some state ρ and some set of states S is small: Find a projector P such that S consists of all states satisfying (b). Then show that with high probability, measuring P would succeed.

Lemma 9 (Convexity of the trace distance) Let $\rho = \sum_i p_i \rho_i$ and $\sigma = \sum_i p_i \sigma_i$ with $\sum_i p_i = 1$, $p_i \geq 0$. Then

$$\text{TD}(\rho, \sigma) \leq \sum_i p_i \text{TD}(\rho_i, \sigma_i).$$

This lemma is sometimes useful because it allows to remove some initial random choices from the analysis

A generalisation of this lemma that does not require the probabilities p_i to be the same in ρ and σ also exists.

Further reading: [NC00, Section 9.2.1].

13 Quantum key distribution

The goal of quantum key distribution (QKD, a.k.a. quantum key exchange) is the following. Two parties Alice and Bob communicate over two kinds of channels. The first channel allows to send classical information and is authenticated (but not secret). The second channel allows to send qubits but is insecure (under the control of the adversary). Alice and Bob want to agree on a secret key by communicating only over these channels such that even a computationally unlimited adversary Eve that eavesdrops on the classical channel and controls the quantum channel cannot learn anything about the key. (But Eve is allowed to disrupt the communication.)

The basic idea of quantum key exchange is the following: If Alice sends to Bob qubits encoded in a random basis (unknown to Eve), then if Eve measures the qubits she will necessarily introduce disturbances. Then Alice and Bob perform some checks on the qubits received by Bob, and if Eve eavesdropped, we may expect some of these checks to fail and Alice and Bob will abort the protocol. Otherwise, Alice and Bob use the transmitted qubits to derive a shared secret key.

There are various desirable properties that a QKD protocol should have:

- *Provable security.* It should be possible to actually prove the security of the protocol. This is a must, otherwise we do not gain much over the classical key exchange protocols.
- *Error tolerance.* The key exchange protocol should work even if the communication channel is noisy (introduces errors). This is difficult because a noisy channel also

introduces disturbances that look similar to those introduced by an eavesdropper. So if Alice and Bob abort whenever there is a disturbance, the protocol will never succeed. If they choose not to abort, Eve may learn some information.

- *Realisability.* The protocol should not need to use a quantum computer. It should be executable using only simple operations like sending polarised photons and measuring the polarisation.
- *Arbitrary distance.* The key exchange protocol should work over an arbitrary distance. In realistic channels, the noise increases with the distance. From some distance on, the noise is too large to make key exchange possible. One solution is to add relays on the way that correct errors or perform other computations, but these relays should not be assumed to be secure (they might be under the control of Eve). Quantum error correction can be used in untrusted relays, but this needs a quantum computer.

The (rough) state of the art is listed in the following table:

	BB84 and others	Lo-Chau	this lecture
Provable security	yes	yes	yes
Error tolerance	yes	yes	no
Realisability	yes	no	no
Arbitrary distance	no	yes	no

Here BB84 and other stands for most of the currently investigated protocols of which (variations of) BB84 [BB84] are the most well-known. Lo-Chau stands for the protocol proposed in [LC99].

In this lecture, we analyse a simplification of the Lo-Chau protocol that does not need to use quantum error correction.

Most research today concentrates on trying to improve the range (distance) of QKD protocol with available technology. Current records lie in the order of 250 km [SWV⁺09], and about 140 km through a wireless connection [SMWF⁺07].

Definition 45 (Security of QKD) *Let a QKD protocol π be given. Let $n \in \mathbb{N}$. Let $\varepsilon > 0$.*

Let an adversary Eve be given (that has full control over the quantum channel between Alice and Bob, but can only listen to but not modify the classical channel between Alice and Bob). Then let $\rho_{ABE}^{\text{Real}} \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ be the density operator describing the joint state of Alice's, Bob's and Eve's system in the case that Alice and Bob do not abort. Here $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^{2^n}$ because Alice's and Bob's final state consist of an n -bit key, and \mathcal{H}_E is some arbitrary Hilbert space defined by Eve.

Let $S_{\text{Ideal}} \subseteq S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ be the set of all states of the form

$$\left(\sum_{k \in \{0,1\}^n} 2^{-n} (|k\rangle\langle k| \otimes |k\rangle\langle k|) \right) \otimes \rho_E, \quad \rho_E \in S(\mathcal{H}_E).$$

By P_{success} denote the probability that Alice and Bob do not abort the protocol and thus output a key (given a particular adversary Eve).

We say that π is ε -secure if the following holds: For every adversary Eve, we have that

$$\exists \rho_{ABE}^{\text{Ideal}} \in S_{\text{Ideal}} : \quad \text{TD}(\rho_{ABE}^{\text{Real}}, \rho_{ABE}^{\text{Ideal}}) \cdot P_{\text{success}} \leq \varepsilon.$$

Intuitively this means that the keys output by Alice and Bob are the same with high probability, that these keys are almost uniformly distributed, and that Eve's information is almost independent of that key.

Definition 46 (Bell states) *The four Bell states are:*

$$\begin{aligned} |\beta_{00}\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\ |\beta_{01}\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\ |\beta_{10}\rangle &= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\ |\beta_{11}\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle \end{aligned}$$

The four Bell states form a basis of \mathbb{C}^4 .

As a shorthand, we write $|\tilde{a}\tilde{b}\tilde{c}\tilde{d}\tilde{e}\tilde{f}\dots\rangle$ for the state $|\beta_{ab}\rangle \otimes |\beta_{cd}\rangle \otimes |\beta_{ef}\rangle \otimes \dots$. In particular, $|\tilde{0}\dots\tilde{0}\rangle = |\beta_{00}\rangle^{\otimes n}$. The states $|\tilde{x}\rangle$ with $x \in \{0,1\}^n$ form a basis of $\mathbb{C}^{2^{2n}}$ ($2n$ -qubit systems).

Lemma 10 *Let a protocol π be given. Let $n \in \mathbb{N}$. Let $\varepsilon > 0$.*

For every adversary Eve, let $\rho_{ABE}^{\text{Real}} \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ be the density operator describing the joint state of Alice's, Bob's and Eve's system in the case that Alice and Bob do not abort.

Assume that for every adversary Eve, there is a state $\rho_{ABE}^{\text{Ideal}}$ of the form $\rho_{ABE}^{\text{Ideal}} = |\tilde{0}\dots\tilde{0}\rangle\langle\tilde{0}\dots\tilde{0}| \otimes \rho_E$ such that⁸

$$\text{TD}(\rho_{ABE}^{\text{Real}}, \rho_{ABE}^{\text{Ideal}}) \cdot P_{\text{success}} \leq \varepsilon$$

where P_{success} is again the probability of not aborting.

Then let π' be the protocol where Alice and Bob execute π and at the end of the protocol additionally measure each of their qubits in the computational basis (and forget the result). Then π' is an ε -secure QKD protocol.

⁸Note that we implicitly reorder the qubits here in the sense that formally, $|\tilde{0}\dots\tilde{0}\rangle\langle\tilde{0}\dots\tilde{0}|$ describes the state of Alice's and Bob's system when we order the qubits as follows: First qubit of Alice, first qubit of Bob, second of Alice, second of Bob, etc. On the other hand, when we wrote $\sum_x (|x\rangle\langle x| \otimes |x\rangle\langle x|)$ we ordered them as follows: First qubit of Alice, second qubit of Alice, etc., first qubit of Bob, second of Bob, etc. This is just done for notational convenience.

By virtue of Lemma 10, we only need to construct a protocol π as in Lemma 10.

In the following, we assume the QKD protocol we are constructing to be of the following form: Alice and Bob both expect qubits from *Eve*. That is, instead of Alice sending the qubits to Bob, both parties expect these qubits from an untrusted source. It is easy to see that if this protocol is secure, then the protocol where Alice produces and sends the qubits is also secure. This simplifies analysis as *Eve*'s strategy can be completely described by the state she sends to Alice and Bob.

The state that Alice and Bob expect from *Eve* is $|\tilde{0}\dots\tilde{0}\rangle$, that is, Alice and Bob expect that the i -th qubit of Alice's system and the i -th qubit of Bob's system together are in the $|\beta_{00}\rangle$ state. Of course, we do not assume that *Eve* indeed sends such a state.

Definition 47 *Let a state $\rho \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ be given with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^m$. Let $q \in \mathbb{N}$, $q \leq m$. The Bell test is the following procedure:*

- Choose q distinct indices $i_1, \dots, i_q \in \{1, \dots, m\}$.
- For each index i , measure the i -th Alice-Bob qubit pair of ρ using one of the following measurements:
 - $P_{yes} := |\beta_{00}\rangle\langle\beta_{00}| + |\beta_{01}\rangle\langle\beta_{01}|$ and $P_{no} := 1 - P_{yes}$. (I.e., we check that the state is not $|\beta_{10}\rangle$ or $|\beta_{11}\rangle$.)
 - $P_{yes} := |\beta_{00}\rangle\langle\beta_{00}| + |\beta_{10}\rangle\langle\beta_{10}|$ and $P_{no} := 1 - P_{yes}$. (I.e., we check that the state is not $|\beta_{01}\rangle$ or $|\beta_{11}\rangle$.)
- If this measurement returned no, abort.

Note that this test cannot be directly implemented by Alice and Bob because it performs measurements on the joint state of Alice and Bob that cannot be implemented locally. On the exercise sheet, however, we devise an equivalent test that can be implemented with local operations and classical communication (the latter will then be performed through the authenticated channel).

If $\rho = |\tilde{0}\dots\tilde{0}\rangle\langle\tilde{0}\dots\tilde{0}|$, then the Bell test passes with probability 1. If $\rho = |\tilde{x}\rangle\langle\tilde{x}|$ where x has more than t bitpairs that are not 00, the Bell test passes with probability at most approximately $(1 - \frac{t+1}{2m})^q$. The exact bound is somewhat more complicated to compute, we just write $\delta_q \approx (1 - \frac{t+1}{2m})^q$ for that bound. Note that for $t = 0$, even for $q = m$, this does not converge to 0, so we cannot use this test to ensure that there are no errors in the state. However, if t is a fixed fraction of m , δ_q converges exponentially fast to 0 for $m, q \rightarrow \infty$.

We fix the following notation:

By $|x|$ we mean the number of bitpairs in x that are not 00.

Let P_{ok} be the projector $\sum_{|x| \leq t} |\tilde{x}\rangle\langle\tilde{x}| \otimes I$ (where I is the identity on *Eve*'s system \mathcal{H}_E). That is, intuitively P_{ok} projects onto states that have at most t wrong qubit pairs. For notational convenience, we write $P_{ok}(\rho) := P_{ok}\rho P_{ok}^\dagger$.

Let T denote the (not trace-preserving) quantum operation describing the Bell test. More precisely, given a state $\rho \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$, $T(\rho) := p\tilde{\rho}$ where $\tilde{\rho}$ is the state

after passing the Bell test and p is the probability of passing the Bell test. Note that $\tilde{\rho} = \frac{T(\rho)}{\text{tr}T(\rho)}$.⁹

Lemma 11 *Let a state $\rho \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ be given with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^m$. Let $q \in \mathbb{N}$, $q \leq m$.*

Let $\tilde{\rho} := \frac{T(\rho)}{\text{tr}T(\rho)}$ (the state after passing the Bell test). Let $P_{\text{success}} := \text{tr}T(\rho)$ (the probability of passing the Bell test).

Then $\text{tr}P_{ok}(\tilde{\rho}) \geq \frac{\text{tr}T(\rho) - \delta_q}{\text{tr}T(\rho)}$. That is, the (hypothetical) test whether $\tilde{\rho}$ indeed has at most t bad qubits will fail with probability at most $\frac{\delta_q}{P_{\text{success}}}$.

(Note: In the present lecture, we have only shown this lemma under some restrictions on the form of ρ . In the next lecture, however, we will see how to get the lemma in its full generality.)

In the following, let t -Error denote the set of states $|\Psi\rangle$ that are a superposition of states $|\tilde{x}\rangle$ with $|x| \leq t$. (In other words, in $|\Psi\rangle$, at most t bad qubit pairs occur.) Formally,

$$t\text{-Error} := \text{span}\{|\tilde{x}\rangle : |x| \leq t\}$$

Using Lemma 8, we immediately get:

Lemma 12 *Let a state $\rho \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ be given with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^m$. Let $q \in \mathbb{N}$, $q \leq m$.*

Let $\tilde{\rho} := \frac{T(\rho)}{\text{tr}T(\rho)}$ (the state after passing the Bell test). Let $P_{\text{success}} := \text{tr}T(\rho)$ (the probability of passing the Bell test).

Then there exists a state $\rho' \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ such that

- $\rho' = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \otimes \rho_i$ where each $|\Psi_i\rangle \in t\text{-Error}$. (In other words, in ρ' , at most t bad qubit pairs occur.)
- $\text{TD}(\tilde{\rho}, \rho') \leq \sqrt{\frac{\delta_q}{P_{\text{success}}}}$ and hence $\text{TD}(\tilde{\rho}, \rho') \cdot P_{\text{success}} \leq \sqrt{\delta_q}$.

Definition 48 (Entanglement purification) (Entanglement) purification is a process that needs only classical communication between Alice and Bob as well as local unitary operations and measurements and that achieves the following:

Assume that Alice and Bob share a state $|\Psi\rangle \in \mathbb{C}^{2l}$ with $|\Psi\rangle \in t\text{-Error}$ (where Alice and Bob each hold half of each Bell pair). Then the result of applying the purification is a state $|\tilde{0} \dots \tilde{0}\rangle \in \mathbb{C}^{2n}$.

Lemma 13 *There are values $\alpha, \beta \in (0, 1)$ such that for sufficiently large n and $l \geq \alpha n$ and $t \leq \beta n$, purification is possible.*

By combining Lemma 12, Lemma 13, and Lemma 10, we get the following result:

⁹This encoding of the Bell test is analogous to $P_{ok}(\rho)$ where also both the post-measurement state and the probability are encoded in the operator $P_{ok}(\rho)$ of trace ≤ 1 .

Lemma 14 Let a state $\rho \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ be given with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^m$. Let $q \in \mathbb{N}$, $q \leq m$. Let \mathcal{P} denote entanglement purification that takes $l := m - q$ qubit pairs and returns n qubit pairs and corrects t errors.

Let $\tilde{\rho} := \frac{T(\rho)}{\text{tr}T(\rho)}$ (the state after passing the Bell test). Let $\tilde{\rho}'$ be the result of removing the q qubits that were measured in the Bell-test. Let $\hat{\rho} := \mathcal{P}(\tilde{\rho}')$ be the result of applying the purification \mathcal{P} to the remaining bits $\tilde{\rho}'$. Let ρ^* be the state resulting from measuring Alice's and Bob's state in the computational basis.

Let $P_{\text{success}} := \text{tr}T(\rho)$ (the probability of not aborting the protocol).

Then there exists a state $\rho' \in S_{\text{Ideal}}$ such that $\text{TD}(\tilde{\rho}, \rho') \leq \sqrt{\frac{\delta_q}{P_{\text{success}}}}$ and hence $\text{TD}(\tilde{\rho}, \rho') \cdot P_{\text{success}} \leq \sqrt{\delta_q}$.

In other words, there is a $\sqrt{\delta_q}$ -secure QKD protocol producing n key bits that needs m qubits of quantum communication.

Note that the Bell test can be implemented with only local operations and classical communication using the method described in Homework 6, Problem 4.

Further reading: [NC00, Section 12.6]. (However, things are a bit vague there.)

13.1 QKD via privacy amplification

In this section, we describe an alternative approach construct and prove a quantum key distribution protocol.

In the previous section, we have used a purification protocol in order to transform a Bell state ρ' with $\leq t$ errors into an ideal state $|\tilde{0} \dots \tilde{0}\rangle \langle \tilde{0} \dots \tilde{0}| \otimes \rho_E$. Unfortunately, purification protocols cannot be realized with today's technology as they need quantum computers.

In some cases, one can transform a protocol using purification into one that does uses some classical post-processing instead. We will not pursue this approach but instead give a more direct approach here.

The idea is to first perform the Bell test. We get a state ρ' with t errors. Then Alice and Bob measure the state ρ' in the computational basis. They get keys K_A and K_B , respectively. However, since ρ' contained t errors, K_A and K_B may not be equal, and Eve may know partial information about K_A or K_B . To make K_A and K_B equal, we apply (classical) error correction, resulting in the same key K . But that leaves the problem that partial information about K is known to Eve (even more now, because she learned something during the error correction). Thus, we need to somehow improve the privacy of K (this is called *privacy amplification*).

To be able to improve the privacy of K , we first need to have a notion to measure that privacy. The notion of choice is *min-entropy*.

Definition 49 (Min-entropy (classical case)) Let K, E be random variables. Then

$$H_\infty(K) := -\log \max_x \Pr[K = x]$$

and

$$H_\infty(K|E) := -\log \sum_y \Pr[E = y] \max_x \Pr[K = x|E = y].$$

The intuition behind this definition is that $2^{-H_\infty(K|E)}$ is the maximum probability that Eve can guess the key K if she has (classical) side information E about the key.

In our setting, the side information that Eve has is quantum. Thus we need to transfer the definition to the quantum case.

Definition 50 (Min-entropy (quantum case)) *Let $\mathcal{H}_K \otimes \mathcal{H}_B \otimes \mathcal{H}_E$ with $\mathcal{H}_K = \mathbb{C}^{\mathbf{K}}$ be a tripartite system. Let ρ be a cqg-state¹⁰ on this system. (\mathcal{H}_K represents the part of the state containing the key, \mathcal{H}_E Eve's part of the system, and \mathcal{H}_B any other parts of the system not belonging to the key or Eve.)*

We define the min-entropy as:

$$H_\infty(K|E)_\rho := -\log \max_{i,M} \Pr[K = M \text{ given } \rho].$$

Here $i \in \mathbb{N}$, and M ranges over arbitrary measurements on $\mathcal{H}_E \otimes \mathbb{C}^i$.

By “ $\Pr[K = M \text{ given } \rho]$ ” we mean the following probability: Measure the system \mathcal{H}_K in the computational basis. Extend the system \mathcal{H}_E by the quantum system $|0\rangle \in \mathbb{C}^i$ (i.e., apply the quantum operation $\rho' \mapsto \rho' \otimes |0\rangle\langle 0|$ to \mathcal{H}_E) and then measure the result using M . Then the outcomes of the two measurements are equal.

Formally, if $M = \{P_x\}_{x \in \mathbf{K}}$, then

$$\Pr[K = M \text{ given } \rho] := \sum_{x \in \mathbf{K}} \text{tr}(|x\rangle\langle x| \otimes I \otimes P_x)(\rho \otimes |0\rangle\langle 0|).$$

where I is the identity on \mathcal{H}_B .

The intuition behind this definition is again that $2^{-H_\infty(K|E)_\rho}$ is the maximum probability that Eve guesses the key K (contained in \mathcal{H}_K) while having access to the system \mathcal{H}_E . Notice that we allow Eve to add an auxiliary system of arbitrary size i containing the state $|0\rangle$. This is because there might be some measurements that Eve wants to perform that need auxiliary qubits.

A useful property of min-entropy is the following chain rule (which also holds for classical min-entropy):

Lemma 15 (Chain rule) *For any density operator ρ , $H_\infty(X|YE)_\rho \geq H_\infty(XY|E)_\rho - \log(|\mathbf{Y}|)$ if Y takes values in \mathbf{Y} .*¹¹

¹⁰A cqg-state means a state that is classical (c) in the first component, and potentially quantum (q) in the second and third. More precisely, $\rho = \sum_x p_i |x\rangle\langle x| \otimes \rho_i$ with $\rho_i \in S(\mathcal{H}_B \otimes \mathcal{H}_E)$.

¹¹This even holds if X and Y are not classical. In this case, \mathbf{Y} is the set of all possible classical states of Y , i.e., Y lives in $\mathbb{C}^{\mathbf{Y}}$. Notice that our definition of H_∞ only allows to talk about classical X, Y , but more general definitions exist [Ren05].

Given a lower bound on the min-entropy of a key K (given Eve's side information), we can then "extract" a fully random key K' from that key K . To do so we use the following tool:

Definition 51 (Strong extractor (classical case)) A function $Ext : X \times R \rightarrow Y$ is a strong (k, ε) -extractor iff the following holds:

Let K, U_R, U_Y, E be random variables. Assume K takes values in X , and U_R and U_Y are uniformly distributed on R and Y , respectively, and U_R and U_Y are independent of all the other random variables.¹² Assume that $H_\infty(K|E) \geq k$. Then

$$SD\left((Ext(K, U_R), U_R, E), (U_Y, U_R, E)\right) \leq \varepsilon.$$

Intuitively, this says that the extracted key $Ext(K, U_R)$ is indistinguishable from a random key U_Y even when given U_R and E , as long as $H_\infty(K|E) \geq k$.

An analogous definition can be given in the quantum case, where we consider the case that Eve has *quantum* side information (but the key K is still classical).

Definition 52 (Strong quantum extractor (quantum case)) A function $Ext : X \times R \rightarrow Y$ is a strong (k, ε) -quantum extractor iff the following holds:

Consider a multi-partite quantum system $\mathcal{H}_K \otimes \mathcal{H}_R \otimes \mathcal{H}_E$ with $\mathcal{H}_K = \mathbb{C}^X$, $\mathcal{H}_R = \mathbb{C}^R$. Consider a ccq-state $\hat{\rho} = (\sum_x p_x |x\rangle\langle x|) \otimes (\sum_r \frac{1}{|R|} |r\rangle\langle r|) \otimes \rho_x$. (In particular, \mathcal{H}_R contains a uniform classical distribution over R .) Assume that $H_\infty(K|E)_{\hat{\rho}} \geq k$.

Let $\hat{\rho}' = \sum_{x,r} p_x |Ext(x, r)\rangle\langle Ext(x, r)| \otimes \frac{1}{|R|} |r\rangle\langle r| \otimes \rho_x$. (I.e., $\hat{\rho}'$ is the result of replacing x by $Ext(x, r)$.)

Let $\hat{\rho}^I = (\sum_r \frac{1}{|Y|} |y\rangle\langle y|) \otimes (\sum_r \frac{1}{|R|} |r\rangle\langle r|) \otimes (\sum_x p_x \rho_x)$. (Here $Ext(x, r)$ is replaced by a uniform $y \in Y$.)

Then $TD(\hat{\rho}', \hat{\rho}^I) \leq \varepsilon$.

Examples for strong extractors are so-called universal hash functions:

Definition 53 A function $f : X \times R \rightarrow Y$ is a universal hash function (UHF) iff for all $x, y \in X$ with $x \neq y$, we have that

$$\Pr[f(x, r) = f(y, r) : r \stackrel{\$}{\leftarrow} R] \leq \frac{1}{|Y|}.$$

Here $r \stackrel{\$}{\leftarrow} R$ means that r is uniformly randomly chosen from R .

Universal hash functions are known to be strong extractors, even in the quantum case:

Lemma 16 (Leftover hash lemma) Let $f : X \times R \rightarrow Y$ be a universal hash function with $|Y| \leq 2^\ell$. Let $k \geq 0$. Let $\varepsilon := 2^{-\frac{1}{2}(k-\ell)-1}$. Then f is a strong (k, ε) -extractor and a strong (k, ε) -quantum extractor.

¹²Formally U_R is independent of (K, U_Y, E) and U_Y is independent of (K, U_R, E) .

Using quantum extractors, we can now construct a quantum key distribution protocol as follows:

- Alice and Bob run the key distribution protocol including the Bell test. After that, they have a state that is close to a state $\rho' = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \otimes \rho_i$ with $|\Psi_i\rangle \in t\text{-Error}$.
- Then Alice and Bob measure their parts of the state, getting keys K and K_B , respectively. (Let the state of the system after that be $\hat{\rho}$.) Then K and K_B will differ in at most t positions, and we get some lower bound $H_\infty(K|E)_{\hat{\rho}} \geq k$ on the min-entropy of Alice's key for suitable k (see below).
- In order to make K and K_B equal, Alice uses a linear code that detects $2t$ errors. Let S be the parity matrix of that code. Alice sends the syndrome $\sigma := S \cdot K$ to Bob (over the authenticated channel). Bob computes a string e with Hamming weight $\leq t$ such that $S(K \oplus e) = \sigma$. The fact that the linear code detects $2t$ errors guarantees that $K \oplus e = K_B$. So Bob knows Alice's key K .
- Eve has now additionally seen σ , so Eve has learned $|\sigma|$ additional bits. But the chain rule (Lemma 15) guarantees that $H_\infty(K|E, \sigma)_{\hat{\rho}} \geq k' := k - |\sigma|$.
- Given a strong (k', ε) -quantum extractor $Ext : \{0, 1\}^n \times R \rightarrow \{0, 1\}^m$ where n is the length of K , Alice picks $r \in R$ and sends r to Bob (over the authenticated channel). Then Alice and Bob compute $K' := Ext(K, r)$. The definition of strong quantum extractors guarantees that the resulting key K is indistinguishable from a uniformly random key (even given r, E, σ).

What we left open in this description is whether $H_\infty(K|E)_{\hat{\rho}} \geq k$ for some suitable k . The following lemma tells us that this is the case:

Lemma 17 *Let $N := |\{x \in \{0, 1\}^{2n} : |x| \leq t\}|$ (remember that $|x|$ denotes the number of bit pairs that are non-zero). Let $\rho' = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \otimes \rho_i$ with $|\Psi_i\rangle \in t\text{-Error}$. Let $\hat{\rho}$ be the result of measuring Alice's and Bob's system in the computational basis in ρ' . Then $H_\infty(K|E)_{\rho'} \geq n - \log N \geq n - t \log 3n$ (here K refers to Alice's system).*

Using this lemma, and appropriate choices for the various parameters, we get a secure quantum key distribution scheme.

14 Quantum Commitments

A commitment protocol is a protocol with two parties, Alice and Bob (or sender and recipient). It consists of two phases, the *commit* and the *open phase* (also known as *unveil phase*). In the commit phase, Alice runs with some input $b \in \{0, 1\}$ and Bob has no input. No output is made. In the open phase, both Alice and Bob have no input. Bob outputs a bit $b' \in \{0, 1\}$ or aborts.¹³ Intuitively, we require that Bob will output the bit

¹³We assume that Alice never aborts and that Bob does not abort in the commit phase. This is possible without loss of generality, since instead of aborting they may just send dummy messages.

b' that Alice committed herself to in the first phase, that is, Alice cannot change her mind about the bit (binding property). On the hand, we do not want Bob to learn the bit b before the open phase (hiding property). In the following, we assume that Alice and Bob are quantum machines and have a quantum channel between them. Since commitments are not supposed to give security against outside adversaries (but rather against the case that Alice or Bob cheats), we do not need any authenticated or secret channels.

Formally, a secure commitment scheme is one that has the properties: correctness, hiding, and binding.

Definition 54 (Correctness) *We call a commitment protocol ε_C -correct if for honest Alice and Bob, and for any Alice-input $b \in \{0, 1\}$, when executing the commit and the open phase, the probability that Bob outputs $b' = b$ in the open phase is at least $1 - \varepsilon_C$.*

Definition 55 (Hiding) *We call a commitment protocol ε_H -hiding if the following holds: Fix some malicious Bob. Let ρ_b be the state of honest Alice's and malicious Bob's system after performing the commit phase with Alice-input b . Then*

$$\text{TD}(\text{tr}_A \rho_0, \text{tr}_A \rho_1) \leq \varepsilon_H.$$

Definition 56 (Binding) *We call a commitment protocol ε_B -binding if the following holds: Fix some machines A, A_0, A_1 . Let P_b be the probability that honest Bob outputs $b' = b$ after interacting with A in the commit phase and A_b in the open phase. Then*

$$P_0 + P_1 \leq 1 + \varepsilon_B.$$

Intuitively, this means that Alice cannot open as b when she learns b after the commit phase. Note that it is always possible to get $P_0 + P_1 = 1$ since A might just randomly choose b with probability P_b and then perform an honest commit.

Lemma 18 (Schmidt decomposition) *Fix some bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and some quantum state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Then there are orthonormal sets of states*

$$\{|\alpha_i\rangle\} \subseteq \mathcal{H}_A \quad \text{and} \quad \{|\beta_i\rangle\} \subseteq \mathcal{H}_B$$

and reals $\lambda_i \geq 0$ with $\sum_i \lambda_i^2 = 1$ such that

$$|\Psi\rangle = \sum_i \lambda_i |\alpha_i\rangle \otimes |\beta_i\rangle.$$

Lemma 19 (Simultaneous Schmidt decomposition) *Fix some bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and two quantum states $|\Psi\rangle, |\tilde{\Psi}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Assume that $\text{tr}_A |\Psi\rangle\langle\Psi| = \text{tr}_A |\tilde{\Psi}\rangle\langle\tilde{\Psi}|$. Then there are orthonormal sets of states*

$$\{|\alpha_i\rangle\} \subseteq \mathcal{H}_A \quad \text{and} \quad \{|\tilde{\alpha}_i\rangle\} \subseteq \mathcal{H}_A \quad \text{and} \quad \{|\beta_i\rangle\} \subseteq \mathcal{H}_B$$

and reals $\lambda_i \geq 0$ with $\sum_i \lambda_i^2 = 1$ such that

$$|\Psi\rangle = \sum_i \lambda_i |\alpha_i\rangle \otimes |\beta_i\rangle.$$

and

$$|\tilde{\Psi}\rangle = \sum_i \lambda_i |\tilde{\alpha}_i\rangle \otimes |\beta_i\rangle.$$

This lemma intuitively states that if two states are indistinguishable when looking only at the second system, then the corresponding vectors $|\beta_i\rangle$ in the Schmidt decomposition can be chosen to be the same for both states.

Lemma 20 (Purification of the commit phase) *Assume a commitment protocol π that is ε_C -correct, ε_H -hiding, and ε_B -binding. Then there is a commitment protocol $\tilde{\pi}$ that is ε_C -correct, ε_H -hiding, and ε_B -binding and that performs only unitary operations during the commit phase (that is, no honest party measures, and no classical channel is used between the parties).*

The basic idea of the transformation underlying Lemma 20 is to do the following:

- First, replace any use of a classical channel by sending the classical bit on a quantum channel, encoded in the computational basis. Here both the sender and the recipient measure the bit before/after sending to ensure that it is classical even if the other party is cheating.
- Second, generate random bits as follows: If a random bit is needed, take a bit in the $|0\rangle$ state, apply Hadamard to it (resulting in the $|+\rangle$ state), and measure the bit in the computational basis.
- Third, replace every measurement (including those introduced in the steps before) by a CNOT that stores the result the measurement would have onto a fresh quantum register. This register is never used again, thus storing the information in this register has the same effect as measuring.

Theorem 5 (Impossibility of quantum bit commitment) *There is no 0-correct 0-binding 0-hiding commitment protocol.*

Note that this theorem does not exclude the possibility of quantum bit commitment under additional assumptions, e.g., if the adversary is computationally bounded, or if it is bounded in the size of its quantum memory (see next section).

As stated, the theorem does not exclude that ε_C -correct ε_H -hiding ε_B -binding commitment protocols might exist for small but non-zero $\varepsilon_C, \varepsilon_H, \varepsilon_B$. A more careful analysis, however, excludes this [May97].

14.1 Bounded quantum storage model

In the bounded quantum storage model, we assume that there is an upper bound n on the amount of quantum memory the adversary can store over a longer period of time (where we do not specify that period of time, but require that between commit and open at least that time passes). In the bounded quantum storage model more is

possible to design secure quantum commitment protocols without having to resort to any unproven computational assumption like the hardness of inverting some function (one-way function).

Consider the following protocol from [DFSS05]:

- Let m be some parameter.
- *Commit phase:* In the commit phase, Bob (the recipient) chooses m bits $x_i \in \{0, 1\}$, and m bits $b_i \in \{0, 1\}$. Then Bob encodes each x_i in a basis specified by b_i ; call the resulting qubit Ψ_i . More precisely, if $b_i = 0$, then x_i is encoded in the computational basis $|0\rangle, |1\rangle$; if $b_i = 1$, then x_i is encoded in the diagonal basis $|+\rangle, |-\rangle$. Then Bob sends the qubits $|\Psi_1\rangle, \dots, |\Psi_m\rangle$ to Alice.

If $c = 0$ (Alice wants to commit to 0), then Alice measures all qubits in the computational basis; if $c = 1$ (Alice wants to commit to 1), then Alice measures all qubits in the diagonal basis. Let the results of these measurements be \tilde{x}_i .

- *Open phase:* Alice sends c and all the bits \tilde{x}_i to Bob. Bob checks whether $x_i = \tilde{x}_i$ for all i with $b_i = c$. If so, Bob outputs $c' := c$. Otherwise, Bob aborts.

Theorem 6 (Commitment in the quantum bounded storage model [DFSS05])

Fix some constant $\delta > 0$. Let n denote the security parameter and assume that n is also the quantum memory bound of the adversary. Assume that in the above protocol, $m \geq (4 + \delta)m$. Then there is a negligible function μ such that the above protocol is perfectly correct and hiding (i.e., 0-correct and 0-hiding) and $\mu(n)$ -binding.

Note that the protocol does not need any quantum storage on the side of Alice and Bob. Current technology does not allow to implement this protocol because it assumes that no errors occur on the quantum channel. However, a straightforward modification in which Bob accepts a certain amount of errors in his check in the open phase can be shown to be secure.

For the proof of Theorem 6, we need a variant of the definition of min-entropy, the smooth min-entropy, the smooth min-entropy. We only give the formal definition in the classical case:

Definition 57 (Smooth min-entropy) Let X and Y be discrete random variables. Let $\varepsilon > 0$. For an event E , let $H_\infty(XE) := -\log \max_x \Pr[X = x \text{ and } E \text{ holds}]$ and let $H_\infty(XE|Y) := \min_y H_\infty(XE|Y = y)$.

The smooth min-entropy $H_\infty^\varepsilon(X)$ is defined as $H_\infty^\varepsilon(X) := \max_E H_\infty(XE)$, and the conditional smooth min-entropy $H_\infty^\varepsilon(X|Y)$ is defined as $H_\infty^\varepsilon(X|Y) := \max_E H_\infty(XE|Y)$, where in both cases E ranges over all events with $\Pr[E] \geq 1 - \varepsilon$.

Intuitively, having smooth min-entropy $\alpha = H_\infty^\varepsilon(X)$ means having min-entropy α except in a rare case $\neg E$ which has probability at most ε .

We omit the definition of smooth min-entropy in the quantum case, see [Ren05].

Theorem 7 (Uncertainty relation) Fix a constant $\lambda > 0$. Let $\rho \in S(\mathbb{C}^{2^m})$. Let $b_1, \dots, b_m \in \{+, \times\}$ be uniformly and independently chosen bases (+ denotes the computational, and \times the diagonal basis). Let x_i denote the result of measuring the i -th bit of ρ in basis b_i . Then there exists a negligible ε such that

$$H_\infty^{\varepsilon(m)}(x_1, \dots, x_m | b_1, \dots, b_m) \geq (\frac{1}{2} - 2\lambda)m.$$

This theorem states that if we are given an arbitrary state ρ (that about which we may know as much as we like), and then a basis is randomly chosen, then we will necessarily have approximately $m/2$ bits of uncertainty about the outcome of measuring the state in this basis. In other words, for no state can we know what would be the outcome of measuring that state for each possible basis. (Although for particular fixed bases, we can certainly know the output: For example, if b is the computational basis and $x = |0\rangle$.)

Theorem 7 is shown in [DFR⁺07] (Corollary 3.4 in the full version).

The chain rule (Lemma 15) also holds for smooth min-entropy:

Lemma 21 (Chain rule [Ren05]) For any density operator ρ , $H_\infty^\varepsilon(X|YE)_\rho \geq H_\infty^\varepsilon(XY|E)_\rho - \log(|\mathbf{Y}|)$ if Y takes values in \mathbf{Y} .¹⁴

Lemma 22 (Min-entropy splitting) Let X_0, X_1, B be random variables. Fix $\varepsilon, \varepsilon' > 0$. Then there exists a function C with range $\{0, 1\}$ such that

$$H_\infty^{\varepsilon+\varepsilon'}(X_{\tilde{C}}|C, B) \geq H_\infty^\varepsilon(X_0, X_1|B)/2 - 1 - \log \frac{1}{\varepsilon}$$

where $C := C(X_0, X_1, B)$ and $\tilde{C} := 1 - C$.

Intuitively, this lemma means that if we have an uncertainty α about X_0, X_1 , then we have uncertainty approximately $\alpha/2$ about X_0 or X_1 . (Where the random variable \tilde{C} indicates which of the X_0, X_1 is currently the uncertain one.)

Lemma 22 is shown in [DFR⁺07] (Corollary 4.3 in the full version).

Lemma 23 Let $\varepsilon > 0$. Let ρ be a cq-state consisting of quantum systems X and E . I.e., we have a state $\rho = \sum_{xb} p_{xb} |x\rangle\langle x| \otimes \rho_x$.

Let \tilde{X} denote the outcome of applying some quantum operation \mathcal{E} to E and then measuring E in some basis. Let X denote the outcome of measuring X in the computational basis.

Then

$$\Pr[X = \tilde{X}] \leq 2^{-H_\infty^\varepsilon(X|E)} + \frac{3}{2}\varepsilon.$$

Notice that for $\varepsilon = 0$, this follows from the definition of min-entropy (Definition 50).

Further reading: For the Schmidt decomposition, see [NC00, Section 2.5]. For the impossibility of quantum commitment, see [May97]. For commitment in the bounded quantum storage model, see [DFSS05] and [DFR⁺07]. For definitions of min-entropy in the quantum case and a lot of results concerning these, see [Ren05]. A proof that the definition of min-entropy as given here is equivalent to that in [Ren05] can be found in [KRS09].

¹⁴This even holds if X and Y are not classical.

15 Revocable quantum time vaults

A time vault TV is, intuitively, a kind of encryption which can be broken in time T , but not in time $\ll T$.

The idea behind this is that one can use a time vault to send messages into the future. If Alice wants Bob to have some information m after time T , but not before that, she sends $TV(m)$ to Bob, and Bob will only be able to compute m after time T .

A physical intuition for a time vault is that of a strong box with a timer. The strong box opens automatically at time T , but cannot be opened earlier.

Mathematically, a time vault is modeled as a probabilistic algorithm TV that takes a message m and returns a time vault $TV(m)$. Additionally, there should be another algorithm Dec that the honest recipient can use to recover m from $TV(m)$.

A time vault should, ideally, have the following security:

Definition 58 (T -hiding time vault) *We call a time vault TV T -hiding if the following holds: For any probabilistic T -time adversary B^* , there is a negligible μ such that for any η , and any m_0, m_1 (in the message space of TV), we have that $|\Pr[P_0 - P_1]|$ is negligible in η where $P_i := \Pr[b = 1 : V \leftarrow TV(m_i), b \leftarrow B^*(1^\eta, V)]$.*

Here T and TV may implicitly depend on the security parameter η . And $x \stackrel{\$}{\leftarrow} A$ means that x that is the output of algorithm A .

We say TV is T -hiding against quantum adversaries if the above holds for all quantum T -time adversaries B^ .*

This means that before time T , no adversary can find out anything about the message m from $TV(m)$ (except with negligible probability).

However, T -hiding time-vaults do not capture all the useful properties that strong-boxes with timers have. Consider as an example the following situation (that might come straight from a movie). Alice has to go to a meeting with criminals. Alice fears that she might not survive the meeting, unless she has some way to pressure the criminals into letting her leave alive. A possible approach is to leave some compromising information about the criminals in the hands of some trusted friend Bob (e.g., evidence photos). If Alice does not return within a day, Bob is supposed to reveal the information to the police. The problem with this is that Alice might not wish Bob to get these photos except if Alice does not return from the meeting safely. Using strongboxes with timers, Alice could achieve her goal: She puts the information into the strongbox, and puts the timer to $T := 1 \text{ day}$. If Alice does not return alive, Bob can open the strongbox after a day and deliver the information to the police. If Alice does return alive before time T , she asks the strongbox back. When she gets it back, she will know that Bob did not keep the information and cannot deliver it to the police or anyone else.¹⁵

What happens if we use (digital) time vaults to implement the above? In this case, there is no way how Bob can give back the time vault to Alice. Of course, he could just give back the bitstring $TV(m)$ to Alice, but that will not convince her of anything,

¹⁵Of course, Bob can just refuse to give the strongbox back. But in that case, Alice will at least know that Bob is cheating her, and this might already be sufficient to deter Bob.

because Bob could have made a copy of $TV(m)$. So Alice will never know whether Bob did not decrypt a copy of $TV(m)$ at his leisure. So, the following property is obviously impossible with classical time vaults:

Definition 59 (Revocably hiding time vaults – informal) *We call a time vault TV T -revocably hiding if there is some algorithm Rev (revocation) such that the following holds (even for malicious Bob): If Alice sends $TV(m)$ to Bob, and Bob sends back some value X after time at most T , and Alice runs $Rev(X)$, and Rev returns 1, then Bob does not learn anything about m later (even when computing considerably more time than T).*

Note that the reason that revocably hiding time vaults are impossible in the classical setting comes from the fact that a malicious Bob can just copy $TV(m)$. So, an obvious idea is to use a quantum state as a time vault. Then, at least, it is not obvious that Bob can just break the protocol by copying $TV(m)$.

The following protocol is a first candidate for a revocable time vault:

Definition 60 (A quantum time vault) *Assume a T -hiding time vault TV that is secure against quantum adversaries. Then the quantum time vault QTV is defined as follows:*

- Upon input a message $m \in \{0, 1\}^n$, pick n random bases $B = B_1 \dots B_n \in \{+, \times\}^n$.
- Encode the bits of m using in the bases B_1, \dots, B_n . We call the resulting state $|m\rangle_B$.
- Compute $V \leftarrow TV(B)$.
- Return $(|m\rangle_B, V)$ (for sending to Bob).

The revocation algorithm Rev is as follows:

- Input: An n -bit quantum register X (supposedly the state $|m\rangle_B$ that Bob returned to Alice).
- Measure X in bases B , call the outcome m' .
- If $m = m'$, Rev returns 1.

Unfortunately, QTV does not guarantee that Bob learns nothing about m . Bob can guess some of the bases B correctly, and thus get some of the bits of m . Thus QTV is not T -revocably hiding in the sense of Definition 59. We can only show the weaker property that Bob cannot guess all of m (for a uniformly random message m) assuming the revocation succeeded. We say that QTV is T -revocably one-way. Strengthening QTV to be T -revocably hiding is ongoing research (but it looks promising). Formally, what we can show is the following:

Lemma 24 *For any pair B_1^*, B_2^* of machines such that B_1^* runs in quantum-time T and B_2^* runs in quantum polynomial-time, there is a negligible μ such that for all η the following holds:*

$$\Pr[m = m^* \text{ and } ok = 1 : m \xleftarrow{\$} \{0, 1\}^\eta, X \leftarrow QTV(m), \\ X' \leftarrow B_1^*(1^\eta, X), ok \leftarrow Rev(X'), m^* \leftarrow B_2^*(1^\eta)] \leq \mu(\eta).$$

Here B_1^*, B_2^* may share state (i.e., whatever B_1^* has stored, B_2^* has later access to).

Proof sketch. The lemma is shown by considering a sequence of “games”. The first represents what happens in the lemma, and all subsequent games are slight modifications of it. We derive the lemma by deriving various facts about the games. Differences between the games are highlighted in blue.

Game 1 (Original protocol)

- (a) $B \xleftarrow{\$} \{+, \times\}^\eta$.
- (b) $V \leftarrow TV(B)$.
- (c) $m \xleftarrow{\$} \{0, 1\}^\eta$.
- (d) Initialize the quantum register X with $|m\rangle_B$.
- (e) Adversary B_1^* gets V, X .
- (f) Measure X in basis B , outcome m' .
- (g) $ok := 1$ iff $m = m'$.
- (h) Adversary B_2^* is run. Returns m^* .

Game 1 describes the situation in the lemma. Thus to show the lemma, we need to show that $\Pr[m = m^* \wedge ok = 1 : \text{Game 1}]$ is negligible. (The notation $\Pr[E : \text{Game } i]$ means the probability that E holds in Game i .)

We now modify the previous game. Instead of picking m at random and initializing X with $|m\rangle_B$, we put Bell pairs into XY , and then measure Y in the basis B to figure out what m is.

Also, instead of testing whether $m = m'$ where m and m' are what we get when measuring X, Y in basis B , we directly measure whether X and Y have the same state in basis B . More formally, we apply to XY the measurement described by the projector P_B^- where $P_B^- = \sum_m |m\rangle\langle m| \otimes |m\rangle\langle m|$.

Game 2 (Using EPR pairs)

- (a) $B \xleftarrow{\$} \{+, \times\}^\eta$.
- (b) $V \leftarrow TV(B)$.
- (c) ~~$m \xleftarrow{\$} \{0, 1\}^\eta$.~~
- (d) Initialize the quantum registers XY with $|\beta_{00}\rangle^{\otimes \eta}$.
- (e) Adversary B_1^* gets V, X .
- (f) ~~Measure X in basis B , outcome m' .~~
- (g) ~~$ok := 1$ iff $m = m'$.~~
- (h) Apply measurement P_B^- to XY . $ok := 1$ if measurement succeeds.
- (i) Measure Y in basis B , outcome m .
- (j) Adversary B_2^* is run. Returns m^* .

We have that

$$\Pr[m = m^* \wedge ok = 1 : \text{Game 1}] = \Pr[m = m^* \wedge ok = 1 : \text{Game 2}] \tag{2}$$

Roughly, the reason is that we also get the state $|m\rangle\langle m|_B$ in X when we initialize XY with Bell pairs and compute x by measuring Y in basis B .

Next, we remove all computations that take more than time T from the game. This will allow us to use the T -hiding property of TV later on. Instead, we apply a test that tells us whether the state of the system is such that Bob (B_2^*) cannot ever guess m . The test that we do is whether XY contains Bell pairs (with up to t errors for suitably chosen t). For this, we apply the measurement described by the projector P_t^{Bell} where P_t^{Bell} is the projector onto t -Error (which in turn is defined in Section 13).

Game 3 (Testing the state)

- (a) $B \xleftarrow{\$} \{+, \times\}^\eta$.
- (b) $V \leftarrow TV(B)$.
- (c) Initialize XY with $|\beta_{00}\rangle^{\otimes \eta}$.
- (d) Adversary B_1^* gets V, X .
- (e) Apply measurement $P_B^=$ to XY . $ok := 1$ if measurement succeeds.
- (f) Apply measurement P_t^{Bell} to XY . $tErr := 1$ if measurement succeeds.
- (g) ~~Measure Y in basis B , outcome m .~~
- (h) ~~Adversary B_2^* is run. Returns m^* .~~

We have the following fact:

$$\Pr[tErr = 0 \wedge ok = 1 : \text{Game 3}] \text{ negl.} \implies \Pr[m = m^* \wedge ok = 1 : \text{Game 2}] \text{ negl.} \quad (3)$$

The reason for this is that $tErr = 1$ only (up to negligible error) if the state in XY is close to a t -error state in Game 3. (Essentially by Lemma 8.) Since up to the measurement P_t^{Bell} , the two games are identical, this implies that also the state in XY in Game 2 is close to a t -error state. However, if XY is close to a t -error state, then B_2^* will have negligible probability of guessing m , the result of measuring Y (assuming suitably chosen t). (This is analogous to showing that in a QKD protocol without entanglement purification/privacy amplification, the probability of guessing the key is negligible. Cf. Lemma 17.)

We can now replace $TV(B)$ by $TV(0)$.

Game 4 (Using fake TV)

- (a) $V \leftarrow TV(0)$.
- (b) Initialize XY with $|\beta_{00}\rangle^{\otimes \eta}$.
- (c) Adversary B_1^* gets V, X .
- (d) $B \xleftarrow{\$} \{+, \times\}^\eta$.
- (e) Apply measurement $P_B^=$ to XY . $ok := 1$ if measurement succeeds.
- (f) Apply measurement P_t^{Bell} to XY . $tErr := 1$ if measurement succeeds.

We have that

$$\left| \Pr[tErr = 0 \wedge ok = 1 : \text{Game 3}] - \Pr[tErr = 0 \wedge ok = 1 : \text{Game 4}] \right| \text{ is negligible.} \quad (4)$$

This is because we have removed the call to B_2^* , and B_1^* runs in time T .¹⁶ By definition of

¹⁶Here we are simplifying by assuming that all the other computation steps in Game 3 take no time at all. A precise proof would take these into account, then we would need that TV is a $(T + T')$ -hiding time vault, where T' is the time spent in these additional computation steps.

T -hiding no T -time algorithm can distinguish between $TV(B)$ and $TV(0)$, in particular not the code in Game 3.

Finally, we have

$$\Pr[tErr = 0 \wedge ok = 1 : \text{Game 4}] \text{ is negligible.} \quad (5)$$

Reason: Picking B at random and then testing whether X, Y are equal in that basis (P_B^-) is essentially a Bell test. (Here it is important that now B is not used before applying P_B^- , otherwise we could not consider it random.) Remember that in the analysis of our QKD protocol, after successfully performing the Bell test, the probability of not passing a measurement that checks if there are at most t errors, is negligible. A very similar computation shows (5).

We can now finish the proof. By (4,5) we have that $\Pr[tErr = 0 \wedge ok = 1 : \text{Game 3}]$ is negligible. By (3), this implies that $\Pr[m = m^* \wedge ok = 1 : \text{Game 2}]$ is negligible. And finally, by (2), this implies that $\Pr[m = m^* \wedge ok = 1 : \text{Game 1}]$ is negligible. Since Game 1 describes the situation in Lemma 24, that lemma follows. \square

How to construct revocably hiding quantum time vaults? The protocol QTV from Definition 60 has the disadvantage that partial information about the message m is leaked. We would like a T -revocably hiding quantum time vault (as in Definition 59). This is ongoing research, we currently have two approaches (that both probably work):

- *Encode m in a larger codeword w such that knowledge about a part of w does not reveal anything about m .* The challenge here is choosing the right encoding and how to extend the proof idea of Lemma 24 to show that indeed no information about m is leaked.
- *Use the hashed message as key.* To send a message m , we first pick a random k , then we send $(QTV(k), H(k) \oplus m)$ where H is a suitable hash function. Even though $QTV(k)$ leaks some information about k , k is sufficiently uncertain that we do not learn anything about $H(k)$. Since nothing is known about $H(k)$, $H(k) \oplus m$ hides m . Probably the security of this scheme can only be shown in the so-called random oracle model. (Meaning the proof is based on a heuristic.)

16 Zero-knowledge proofs

A zero-knowledge proof is, intuitively speaking, a protocol in which a prover P is able to convince a verifier V of the truth of a statement x in such a way that the verifier learns nothing (except, of course, the fact that x is true).

More formally, we first fix a relation R . If $(x, w) \in R$, we say that w is a witness for the statement x . We defined the language L_R of true statements as follows:

$$L_R := \{x : \exists w.(x, w) \in R\}.$$

In other words, $x \in L_R$ iff there is a witness for x .

We first define what it means for (P, V) to form a proof system (in the classical case). For this, we first introduce the following notation: For two machines A, B , $\langle A(a), B(b) \rangle$ denotes the output of B after an interaction of A and B where A gets input a and B gets input b .

Definition 61 (Proof systems) We call a pair (P, V) of interactive machines a proof or proof system for the relation R with soundness-error ε iff the following two conditions are fulfilled:

- Completeness: For any $(x, w) \in R$, we have that $\Pr[\langle P(x, w), V(x) \rangle = 1] = 1$. (I.e., when the prover gets a valid witness w for x , then he manages to convince V of the truth of x .)¹⁷
- Soundness: For any (potentially computationally unlimited) machine P^* , and for any $x \notin L_R$, we have $\Pr[\langle P^*(\cdot), V(x) \rangle = 1] \leq \varepsilon$. (I.e., except for probability ε , no prover can convince V of a wrong statement x .)

We can now define what it means that the verifier does not learn anything:

Definition 62 (Zero-knowledge) A pair (P, V) of interactive machines is statistical zero-knowledge if for any polynomial-time¹⁸ V^* there exists a polynomial-time S and a negligible μ such that for all $(x, w) \in R$ and all $z \in \{0, 1\}^*$, we have

$$\text{SD}(\langle P(x, w), V^*(x, z) \rangle, S(x, z)) \leq \mu(|x|).$$

(I.e., the simulator can simulate anything V^* learns without knowing the witness w .)

An example for a zero-knowledge proof is the following:

Definition 63 (Graph isomorphism) The relation R_{GI} is defined as follows: $(x, w) \in R_{GI}$ iff $x = (G_1, G_2)$ and $w = \phi$ where G_1, G_2 are graphs and $\phi : G_1 \rightarrow G_2$ is a graph isomorphism.

Definition 64 (Graph isomorphism proof system) Let GIP denote the following protocol between machine P and V :

- P gets inputs $x = (G_1, G_2)$ and $w = \phi$.
- V gets input x .
- P picks a uniformly random permutation ψ_1 on the vertices of G_1 and computes $H := \psi_1(G_1)$. (Notice that now $\psi_1 : G_1 \rightarrow H$ is an isomorphism.)
- P sends H to V .
- V picks $i \in \{1, 2\}$ uniformly and sends i to P .
- P computes $\psi_2 := \psi_1 \circ \phi^{-1}$ and sends $\psi_i : G_i \rightarrow H$.
- V checks whether $\psi_i : G_i \rightarrow H$ is an isomorphism. If so, V outputs 1.

¹⁷Of course, one could also relax this condition and allow a certain error in the completeness instead of requiring probability 1. For simplicity, we stick to the present definition.

¹⁸In this section, we will call a machine polynomial-time if its running-time is bounded by a polynomial in the length of its first argument.

Theorem 8 *GIP is a statistical zero-knowledge proof system.*

We now present the definitions of zero-knowledge proofs for the quantum case. For two quantum or classical machines A, B , $\langle A(a), B(b) \rangle$ denotes the quantum state of B (or, if B is classical, its output) after an interaction of A and B where A gets input a and B gets input b . Here a and b may be classical values or density operators.

The definition of being a proof system (i.e., completeness and soundness) is word for word the same as in the classical case (Definition 61), except that P^* is allowed to be a quantum machine.

More interesting is the definition of statistical quantum zero-knowledge:

Definition 65 (Quantum zero-knowledge) *A pair (P, V) of interactive machines is statistical quantum zero-knowledge if for any polynomial-time quantum-machine V^* there exists a polynomial-time quantum-machine S and a negligible μ such that for all $(x, w) \in R$ and all density operators ρ , we have*

$$\text{TD}(\langle P(x, w), V^*(x, \rho) \rangle, S(x, \rho)) \leq \mu(|x|).$$

(I.e., the simulator can simulate anything V^* learns without knowing the witness w .)

Note that in this case, the “auxiliary input” that V^* gets (called z in the case of Definition 62) is a quantum state.

To show that GIP is statistical QZK, we need to construct a suitable simulator S . However, it turns out that the construction from the classical case does not directly carry over. The reason is that the simulator in the classical case uses rewinding: It tries to produce a simulation, and, if it fails, it tries again. In the quantum case, trying again is not necessarily an option, because the first try may have destroyed the input state ρ , so the second try will fail.

What does work, however, is constructing a polynomial-time simulator S_1 that tries to produce a simulation and either produces a perfect simulation or aborts, and that aborts with probability exactly $\frac{1}{2}$. (More precisely, if $(x, w) \in R$ and ρ' is the state output by the simulator $S_1(x, \rho)$, then $\text{tr } P_{\perp} \rho' = \frac{1}{2}$ and $P_{\perp} \rho' P_{\perp} / \text{tr } P_{\perp} \rho' P_{\perp} = \langle P(x, w), V^*(x, \rho) \rangle$ where P_{\perp} projects on the state denoting abort.)

The construction of this simulator is analogous to the classical case and the proof that it produces a perfect simulation with probability $\frac{1}{2}$ also follows very closely the lines of the proof in the classical case.

To produce a simulator S in the sense of Definition 65 from S_1 , we cannot directly follow the classical proof. Instead, we use the following lemma:

Lemma 25 (Quantum rewinding lemma [Wat09]) *Let Q be a unitary operation from $\mathcal{H}_{in} \otimes \mathcal{H}_{anc}$ to $\mathcal{H}_{out} \otimes \mathcal{H}_{succ}$ with $\mathcal{H}_{succ} = \mathbb{C}^2$. (This implies that $\dim \mathcal{H}_{in} \otimes \mathcal{H}_{anc} = \dim \mathcal{H}_{out} \otimes \mathcal{H}_{succ}$ since a unitary operation is a square matrix.)*

Assume that there is a value $p \leq \frac{1}{2}$ such that for any $|\Psi\rangle \in \mathcal{H}_{in}$, we have that applying Q to $|\Psi\rangle \otimes |0\rangle$ and then measuring \mathcal{H}_{succ} in the computational basis gives outcome 1 (success) with probability p (not $\geq p$). Let $|\check{\phi}_{succ}\rangle$ denote the post measurement state in \mathcal{H}_{out} in that case.

Consider the following algorithm (depending on a parameter q):

1. Let $|\Psi\rangle$ denote the input of the algorithm (in \mathcal{H}_{in})
2. Initialize \mathcal{H}_{anc} with $|0\rangle$.
3. Apply Q .
4. Measure \mathcal{H}_{succ} in the computational basis.
5. If the outcome is 1, exit (successfully).
6. Apply Q^\dagger .
7. Apply FLIP to \mathcal{H}_{anc} where $\text{FLIP } |0\rangle := |0\rangle$ and $\text{FLIP } |x\rangle := -|x\rangle$ for $x \neq 0$.
8. Go to 3. (But at most q times.)

Then for a suitable $q \in \text{poly}(1/p)$, we have that

- The probability that R exits successfully is overwhelming.
- The post measurement state in \mathcal{H}_{out} in that case is $|\tilde{\phi}_{succ}\rangle$.

This lemma can be used to construct the simulator S from S_1 : First, we purify S_1 (i.e., replace measurements by CNOTs on ancilla qubits in \mathcal{H}_{anc} initialized with $|0\rangle$), resulting in Q . Then S runs R and outputs the state $|\tilde{\phi}_{succ}\rangle$.

Notice that in the classical case, it is sufficient that S_1 succeeds with probability $\geq p$ (possibly dependent on the auxiliary input z), while in the quantum case, we need that the simulator S_1 succeeds with a probability p that is independent of the auxiliary input ρ .

Notice further that the above lemma only covers the case where the simulation is perfect. There is a variant of that lemma which also covers the case where S_1 produces a state that has negligible trace distance from $\langle P(x, w), V^*(x, \rho) \rangle$. This allows to cover a wider range of protocols and even protocols that are only computationally QZK.

Further reading: An overview over zero-knowledge proofs in the classical case can be found in [Gol01, Chapter 4]. For quantum zero-knowledge, see [Wat09].

17 Factoring

Note: The following section will contain only a simplified exposition that is not complete but will give the rough idea of how to factor integers using quantum computers.

Definition 66 (Factoring problem) *Given a non-prime integer $m > 1$, find an integer $d \mid m$ with $d \neq 1$, $d \neq m$ (a non-trivial divisor).*

Definition 67 (Order finding problem) *Let G be a (multiplicative) group. Given $a \in G$, find the smallest $r > 0$ such that $a^r = 1$ in G . This value r we denote $\text{ord } a$.*

Lemma 26 (Reducing factoring to order-finding) *Given an oracle that solves the order finding problem in groups $G = \mathbb{Z}_m^\times$ (for arbitrary $m > 1$), we can solve the factoring problem with probability at least $\frac{1}{4}$ in polynomial-time using a single query to the order-finding oracle.*

The idea of the reduction is to compute $\gcd(x^{r/2} + 1, m)$ and $\gcd(x^{r/2} - 1, m)$ for random x . With probability at least $\frac{1}{4}$ one of the two gcds will be a non-trivial divisor of m .

Definition 68 (Discrete Fourier transform) *The discrete Fourier transform (DFT) is a linear transformation on \mathbb{C}^N represented by the matrix $D_N = 2^{-N/2} ((e^{2i\pi kl/N}))_{kl} \in \mathbb{C}^{N \times N}$.*

Note that since $2i\pi kl/N$ is an imaginary number, all entries of D_N have absolute value 1.

Lemma 27 (Properties of the discrete Fourier transform)

- *The discrete Fourier transform D_N is unitary.*
- *Frequency analysis: Given a vector x which is p -periodic (i.e., $x_i = x_{i+p \bmod N}$ for all i ; a special case would be a vectors with 1's at every p -th position), $D_N x$ has entries (non-zero values) on the multiples of N/p .¹⁹ Note that N/p intuitively represents the frequency of x .*

Theorem 9 (Realising the discrete Fourier transform) *There is a quantum algorithm taking an n qubit state $|\Psi\rangle$ as input and returning $D_N|\Psi\rangle$ where D_N is the Fourier transform on \mathbb{C}^N with $N = 2^n$. This algorithm runs in polynomial time in n .*

Theorem 10 (Order-finding) *Assume a group G in which exponentiation is feasible in polynomial-time. There is a polynomial-time quantum algorithm that returns $\text{ord } a$ on input of $a \in G$.*

The algorithm roughly goes as follows: Let $a \in G$. Let $N = 2^n$ be sufficiently larger than $|G|$. The algorithm starts with a quantum state $|0\rangle|0\rangle \in \mathcal{H}_X \otimes \mathcal{H}_Y$, the first system $\mathcal{H}_X := \mathbb{C}^N$ encoding integers $\{0, \dots, N-1\}$, and the second system \mathcal{H}_Y encoding group elements of G . It applies the Hadamard transform to every qubit of \mathcal{H}_X . This results in the state $|\Psi_1\rangle \propto \sum_{x \in \{0,1\}^n} |x\rangle|0\rangle$ (\propto means equal up to a normalization factor). We can implement the unitary transformation U that takes $|x\rangle|y\rangle$ to $|x\rangle|y \oplus a^x\rangle$. By applying U to $|\Psi_1\rangle$, we get $|\Psi_2\rangle \propto \sum_{x \in \{0,1\}^n} |x\rangle|a^x\rangle$. We then measure the system \mathcal{H}_B in the computational basis. This results in a measurement outcome $o = a^{x'}$ for some x' . The state after this measurement is $|\Psi_3\rangle \propto \sum_a |a\rangle$ where the sum ranges over all a with $a^x = o = a^{x'}$, i.e., $x = x' + k \text{ord } a$ for some $k \in \mathbb{Z}$. Hence $|\Psi_3\rangle$ is $\text{ord } a$ -periodic. Thus, if we apply the Fourier transform D_N , we get a vector $D_N|\Psi_3\rangle$ which has entries on multiples of $N/\text{ord } a$ (approximately). If we measure the system in the computational basis, we get a multiple of $N/\text{ord } a$. From this we can compute an approximate divider of $\text{ord } a$. Additional work needs to be done to recover the exact value of $\text{ord } a$ from this, but this is a classical computation and omitted here.

¹⁹If $p \nmid N$, this holds only approximately. In this exposition, we will not formulate exact bounds for the approximation.

Definition 69 (Discrete logarithm problem) Let G be a (multiplicative) group and g a generator. Given $y \in G$, find x with $g^x = y$. (That value x is called the discrete logarithm $\text{dlog}_g y$ of y .)

Theorem 11 Assume a group G with generator g in which exponentiation is feasible in polynomial-time. There is a polynomial-time quantum algorithm that returns $\text{dlog}_g a$ on input of $a \in G$.

Further reading: [NC00, Sections 5.1–5.3]

18 Quantum money

Real-world cash (implemented in terms of coins and bank notes) has the following properties:

- *Issued by bank.* There is an entity (called the *bank*) that can produce pieces of cash (called “coins” in the following for simplicity).
- *Unforgeability.* No-one besides the bank can produce more coins. Specifically, we wish that when given q valid coins, it is not possible to make $q + 1$ valid coins out of it.
- *Transferability without involving the bank.* The current owner of a coin can pass on that coin to another party without needing to interact with the bank in the process.
- *Public verifiability.* Anyone can, given a coin, check whether it is indeed a coin (without involving the bank). With real-world cash, this is done by ensuring that there are a number of publicly known but hard to reproduce features that a coin/banknote should have.
- *Anonymity.* When performing transactions (transferring coins), the identity of the involved parties does not have to be involved.

Notice that this only applies to cash. Money that is stored on a bank account does not have the transferability property, since we have to involve the bank to transfer the money to another account. Also, verification involves the bank here, too. (Anonymity typically is not given either, but could be implemented.)

Instead of physical coins and banknotes, we are interested in a cryptographic solution that satisfies the above four properties. With classical cryptography, this is impossible: If A has a coin c , then A can run the transfer protocol to give c to B . Then B has c . However, since A can, before running the transfer protocol, make a copy of his state, A will also still have c . Thus two coins c exist now, violating unforgeability. With quantum cryptography, this attack can be avoided. If the coin c involves a quantum state, then copying c might not be possible, thus after transferring c to B , A will not possess c any more (hopefully).

18.1 Wiesner's protocol

The first quantum money scheme was proposed by Wiesner already in the 1970s [Wie83]. That scheme does not have public verifiability, but its advantage is that it is quite simple to define and to analyze.

Definition 70 (Wiesner's quantum money) *Let η be the security parameter.*

- Issuing money. *The bank picks $s, x \xleftarrow{\$} \{0, 1\}^\eta$, $B \xleftarrow{\$} \{+, \times\}^\eta$. We call s the serial number. Then the bank encodes $|\Psi\rangle := |x\rangle_B$. (I.e., the i -th qubit of $|\Psi\rangle$ is x encoded in basis B .) It issues the coin $(s, |\Psi\rangle)$ and stores (x, B) in its database at index s .*
- Transferring money. *To transfer a coin $(s, |\Psi\rangle)$ to another party, one just sends $(s, |\Psi\rangle)$.*
- Verification (by the bank). *Given a coin (s, C) where C is a quantum register,²⁰, then the bank looks up (x, B) in the database at index s , measures C in bases B , and checks whether the outcome is x . If the database lookup and the check succeed, the bank accepts the coin. (Note: measuring in B and comparing with x is the same as measuring using the projector $|\Psi\rangle\langle\Psi|$ with $|\Psi\rangle := |x\rangle_B$.)*

We do not formalize the notion of unforgeability of quantum money here, but the following lemma states the essence of the unforgeability property of Wiesner's quantum money:

Lemma 28 *Fix an adversary A . Let Bank denote the bank's issuing algorithm, we write $C, s, x, B \leftarrow \text{Bank}$ to denote that C contains the quantum part of the coin, s is the serial number, and x, B the information stored by the bank. Let Ver be the verification routine, we write $ok \leftarrow \text{Ver}(C, s, x, B)$ to denote the verification of (s, C) with when x, B are the information stored by the bank for serial number s . We leave the security parameter implicit.*

Then

$$\Pr[ok_1 = 1 \text{ and } ok_2 = 1 : (C, s, x, B) \leftarrow \text{Bank}, (C_1, C_2) \leftarrow A(C, s), \\ ok_1 \leftarrow \text{Ver}(C_1, s, x, B), ok_2 \leftarrow \text{Ver}(C_2, s, x, B)]$$

is negligible in η .

The proof of this fact uses techniques we already know from the analysis of quantum key distribution and quantum time vaults.

Attacking Wiesner's scheme. However, besides the fact that it does not support verification, Wiesner's scheme has the problem that it is susceptible to a forgeability attack by an adversary that has access to a bank that verifies coins for him. (And such a public service should exist if Wiesner's money is make sense.) More specifically, assume

²⁰We do not write $|\Psi\rangle$ instead of C here, because that notation would imply that C is not entangled with other qubits the adversary may hold.

that bank will, given some coin (s, C) submitted by A , perform the verification, tell A whether verification succeeded, and then return (s, C) (i.e., the post-measurement-state after measuring whether verification succeeded). Then A can produce a copy of the coin (s, C) as follows:

- Denote the η qubits of C by C_1, \dots, C_η .
- For $i = 1, \dots, \eta$ do:
 - Repeat until x_i, B_i are determined:
 - Pick $\hat{x}_i \xleftarrow{\$} \{0, 1\}$, $\hat{B}_i \xleftarrow{\$} \{+, \times\}$.
 - Send $(s, C_1 \dots C_{i-1} |\hat{x}_i\rangle_{\hat{B}_i} C_{i+1} \dots C_\eta)$ to the bank for verification. Note that even in case of a failed verification, the verification measurement does not change the states of C_1, \dots, C_η .
 - If verification fails, \hat{x}_i, \hat{B}_i are not the correct values for x_i, B_i .
 - As soon as all but one value of (\hat{x}_i, \hat{B}_i) are excluded in this way, x_i, B_i are determined.
- Now $x = x_1 \dots x_\eta$ and $B = B_1 \dots B_\eta$ are determined and A can produce arbitrarily many valid coins $(s, |x\rangle_B)$.

Note that this attack only works if the bank hands back the coin even if verification fails. If we require the bank to destroy the coin upon failed verification, I do not know whether the scheme is secure.

18.2 Aaronson-Christiano quantum money.

In [AC12], Aaronson and Christiano propose a quantum money scheme with public verification. To define this scheme, we first introduce some notation:

Let \mathbb{F}_2 be the finite field of size 2. (I.e., elements of \mathbb{F}_2 are essentially bits, with $+$ being the XOR operation, and \cdot being normal multiplication.) Then \mathbb{F}_2^n is a vector space (over \mathbb{F}_2). For a subspace $A \subseteq \mathbb{F}_2^n$, let $|A\rangle := |A|^{-1/2} \sum_{x \in A} |x\rangle$. Furthermore, let A^\perp be the orthogonal complement of A , i.e., $A^\perp = \{y \in \mathbb{F}_2^n : \forall x \in A. x \cdot y = 0\}$ where $x \cdot y$ is the inner product of x and y . Given polynomials p_1, \dots, p_m , let P_p denote the projector $\sum_x |x\rangle\langle x|$ where x ranges over all $x \in \mathbb{F}_2^n$ with $p_1(x) = \dots p_m(x) = 0$.

Definition 71 (Aaronson-Christiano quantum money.) *The scheme is parametrized by n, d, m . Here d could be small (e.g., $d = 4$) and n, m would typically grow linearly with the security parameter.*

- Setup. *The bank picks a public/private key pair for a quantum secure signature scheme and publishes the public key.*
- Issuing money. *The bank picks a random subspace $A \subseteq \mathbb{F}_2^n$ of dimension $n/2$. (This can be done with high probability by picking $n/2$ random elements of \mathbb{F}_2^n and use them as a basis for A .)*

Then the bank picks m uniformly random multivariate polynomials p_1, \dots, p_m in n variables and of degree d such that $p_i(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in \mathbb{F}_2^n$. (Essentially, these p_i serve to describe the space A in a way that does not allow an adversary to recover A efficiently. There is an $O(n^d)$ -time algorithm for picking the polynomials.)

Then the bank picks m uniformly random multivariate polynomials q_1, \dots, q_m in n variables and of degree d such that $q_i(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in \mathbb{F}_2^n$. (The q_i describe A^\perp which is redundant in principle because the p_i determine A and A determines A^\perp . But for efficient verification, we will need q_i , too.)

The bank produces a signature σ on $(p_1, \dots, p_m, q_1, \dots, q_m)$. Finally, it returns the coin $(|A\rangle, p_1, \dots, p_m, q_1, \dots, q_m, \sigma)$.

- Transferring money. To transfer a coin $(|A\rangle, p_1, \dots, p_m, q_1, \dots, q_m, \sigma)$ to another party, one just sends $(|A\rangle, p_1, \dots, p_m, q_1, \dots, q_m, \sigma)$.
- Verification (by anyone knowing the bank's public key). Given a coin $(C, p_1, \dots, p_m, q_1, \dots, q_m, \sigma)$ where C is a quantum register, the verifier first verifies whether σ is a valid signature on $(p_1, \dots, p_m, q_1, \dots, q_m)$. (This ensures that $(p_1, \dots, p_m, q_1, \dots, q_m)$ are indeed issues by the bank and describe a valid coin.) To check if C indeed contains $|A\rangle$ with $A = \{x \in \mathbb{F}_2^n : \forall i. p_i(x) = 0\}$ (as it should if it is a valid coin), the bank measures C with P_p , then applies $H^{\otimes n}$, then measures C with P_q , then applies $H^{\otimes n}$.

Verification succeeds if the signature check and the two measurements succeed.

To see that verification actually succeeds, we need the following facts: The projector implemented by the measurements in the verification is $H^{\otimes n} P_q H^{\otimes n} P_p =: P$. First, for suitably chosen parameters n, m, d , with overwhelming probability we have that $x \in A$ iff $\forall i. p_i(x) = 0$, and that $x \in A^\perp$ iff $\forall i. q_i(x) = 0$. Thus $P = H^{\otimes n} P_{A^\perp} H^{\otimes n} P_A$ where $P_A = \sum_{x \in A} |x\rangle\langle x|$ and P_{A^\perp} analogous. Furthermore, it can be shown that $H^{\otimes n}|A\rangle = |A^\perp\rangle$. Thus $P|A\rangle = H^{\otimes n} P_{A^\perp} H^{\otimes n} P_A|A\rangle = H^{\otimes n} P_{A^\perp} H^{\otimes n}|A\rangle = H^{\otimes n} P_{A^\perp}|A^\perp\rangle = H^{\otimes n}|A^\perp\rangle = |A\rangle$. Thus honest coins pass verification with overwhelming probability. In fact, one can show that $P = |A\rangle\langle A|$, so only honest coins pass verification.

Furthermore, [AC12] shows that the scheme is unforgeable given some (strong) assumption about the hardness of finding solutions of multivariate polynomials.

Further reading: See [Wie83] for Wiesner's original proposal and [MVW12] for a security proof of that scheme. See [AC12] for the Aaronson-Christiano scheme.

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ZK, *see* zero-knowledge

Symbol index

$\text{span } M$	Vector space spanned by vectors in M	
\mathbb{C}	Complex numbers	
\mathbb{Z}	Integers	
\mathbb{N}	Natural numbers, excluding 0	
$ x $	Absolute value of x	
\mathcal{H}	Usually denotes a Hilbert space	
$\text{im } P$	Image of the linear transformation P	
$\langle \Phi, \Psi \rangle$	Inner product of Φ and Ψ	3
x^*	Complex conjugate of $x \in \mathbb{C}$	3
$\ x\ $	Norm (length) of a vector x	3
M^\dagger	Conjugate transpose of matrix M	4
$ \Psi\rangle$	Dirac notation; represents a vector with name Ψ	4
$\langle \Psi $	Dirac notation; the dual of $ \Psi\rangle$	4
$\langle \Phi \Psi\rangle$	Inner product of $ \Phi\rangle$ and $ \Psi\rangle$ (same as $\langle \Phi, \Psi \rangle$)	4
$\text{tr } M$	Trace of a matrix M	4
\otimes	Tensor product	5
H	Hadamard matrix/gate	6
X	Bit flip matrix/gate	7
R_θ	Rotation matrix/gate; rotation angle θ	7
S	Phase shift matrix/gate	7
S_θ	Phase shift matrix/gate; angle θ	7
CNOT	Controlled NOT gate	10
SWAP	SWAP gate	11
$C(U)$	Controlled- U gate	11
$S(\mathcal{H})$	Density operators over \mathcal{H}	15
tr_B	Partial trace (“tracing out B ”)	16
\mathcal{E}	Usually denotes a quantum operation	17
\mathcal{F}	Usually denotes a quantum operation	17
$\text{SD}(X, Y)$	Statistical distance between X and Y	18
$\text{TD}(\rho, \sigma)$	Trace distance between ρ and σ	19
ρ_{ABE}^{Real}	State after execution of QKD protocol (case: no abort)	21
$\rho_{ABE}^{\text{Ideal}}$	Ideal state after execution of QKD protocol	21
S^{Ideal}	Set of all ideal states (after execution of QKD protocol)	21
$ \beta_{ab}\rangle$	Bell state (ab determines which one)	22
$t\text{-Error}$	Set of Bell states with $\leq t$ errors	24
$H_\infty(K)$	Min-entropy of K	25
$H_\infty(K E)$	Conditional min-entropy of K given E	25
$H_\infty(K E)_\rho$	Conditional quantum min-entropy of K given E	26
$x \stackrel{\$}{\leftarrow} Y$	x is uniformly randomly chosen from set Y	27
$H_\infty^\varepsilon(X)$	Smooth min-entropy	31
$TV(m)$	Classical time vault containing m	33
$x \leftarrow A$	x is output from algorithm A	33
Rev	Revocation algorithm for time vault	34
QTV	A quantum time vault 52	34
P_B^-	Measurement whether X, Y are equal in basis B	35
P_t^{Bell}	Measurement whether at most t error in Bell pairs	36
L_R	Language for the relation R	37
$\langle A(a), B(b) \rangle$	B 's output after interacting with A	38
R_{GI}	Relation for graph isomorphism	38
GIP	Proof system for graph isomorphism	38
$\text{ord } a$	Order of group element a	40
D_N	Discrete Fourier transform of size N	41
\propto	Proportional to	41