Introduction

- \( G = (V, E) \)
  - \( V \) = vertex set (nodes)
  - \( E \) = edge set (arcs)
- Graph representation
  - Adjacency list
  - Adjacency matrix
- Graph search
  - Breadth-first search (BFS)
  - Depth-first search (DFS)
    - Topological sort
    - Strongly connected components

Road Networks

In modeling a road network, the vertices may represent the cities or junctions, certain pairs of which are connected by roads/edges.

Graphs

Graphs are one of the unifying themes of computer science. A graph \( G = (V, E) \) is defined by a set of vertices \( V \), and a set of edges \( E \) consisting of ordered or unordered pairs of vertices from \( V \).
**Electronic Circuits**

In an electronic circuit, with junctions as vertices as components as edges.

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**A Simple Metabolic Pathway**

A Simple Metabolic Pathway

**Reference:** On the history of the transportation and maximum flow problem Questioning in graph theory (1973), 3-25

Find a shortest path from station A to station B.
- Need serious thinking to get a correct algorithm.

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**Soviet Rail Network, 1955**

Evolutionary relationship among organisms based on similarity of their cytochrome c proteins

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**Bacteria**

Common ancestral community of primitive cells

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**Skype**

Image source: BBC/Spark/Powerhouse

Green arrows - upregulation
Red arrows - downregulation
Thickness of arrow represents certainty of direction (up/down)

A complete graph

Filter
• Choose a list of genes
(MATING, marked in red)
• Filter for these genes plus
neighbouring genes from the graph

Mutation network $\Delta_{\text{mut}}$
Graphs

- Set of nodes \(|V| = n\)
- Set of edges \(|E| = m\)
  - Undirected edge/graph: pairs of nodes \((u,v)\)
  - Directed edge/graph: pairs of nodes \((u,v)\)
- Set of neighbors of \(v\): set of nodes connected by an edge with \(v\) (directed: in-neighbors, out-neighbors)
- Degree of a node: number of its neighbors (in-degree, out-degree)
- Path: a sequence of nodes such that every two consecutive nodes constitute an edge
- Length of a path: number of nodes minus 1
- Distance between two nodes: the length of the shortest path between these nodes
- Diameter: the longest distance in the graph
Choose

- The boss wants to produce programs to solve the following two problems.
  - **Euler circuit problem:**
    - given a graph $G$, find a way to go through each edge exactly once.
  - **Hamilton circuit problem:**
    - given a graph $G$, find a way to go through each vertex exactly once.
- The two problems seem to be very similar.
- Person A takes the first problem and person B takes the second.
- **Outcome:** Person A quickly completes the program, whereas person B works 24 hours per day and is fired after a few months.

**Euler Circuit: The original Konigsberg bridge**

- Every vertex of this graph has an even degree, therefore there exists an Eulerian graph. Following the edges in alphabetical order gives an Eulerian circuit/cycle.
A joke (continued):

- **Why?** no body in the company has taken Algorithmics class.
- **Explanation:**
  - Euler circuit problem can be easily solved in polynomial time.
  - Hamilton circuit problem is proved to be **NP-hard**.
  - So far, no body in the world can give a polynomial time algorithm for a NP-hard problem.
  - Conjecture: there does not exist polynomial time algorithm for this problem.

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**Flavors of Graphs**

The first step in any graph problem is determining which flavor of graph you are dealing with. Learning to talk the talk is an important part of walking the walk. The flavor of graph has a big impact on which algorithms are appropriate and efficient.
**Directed vs. Undirected Graphs**

A graph \( G = (V, E) \) is undirected if edge \((x, y) \in E\) implies that \((y, x) \) is also in \( E \).

Road networks between cities are typically undirected. Street networks within cities are almost always directed because of one-way streets. Most graphs of graph-theoretic interest are undirected.

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**Weighted vs. Unweighted Graphs**

In weighted graphs, each edge (or vertex) of \( G \) is assigned a numerical value, or weight.

The edges of a road network graph might be weighted with their length, drive-time or speed limit. In unweighted graphs, there is no cost distinction between various edges and vertices.

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**Simple vs. Non-simple Graphs**

Certain types of edges complicate the task of working with graphs. A self-loop is an edge \((x, x)\) involving only one vertex.

An edge \((x, y)\) is a multi-edge if it occurs more than once in the graph.

Any graph which avoids these structures is called simple.

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**Sparse vs. Dense Graphs**

Graphs are sparse when only a small fraction of the possible number of vertex pairs actually have edges defined between them.

Graphs are usually sparse due to application-specific constraints. Road networks must be sparse because of road junctions. Typically dense graphs have a quadratic number of edges while sparse graphs are linear in size.

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**Cyclic vs. Acyclic Graphs**

An acyclic graph does not contain any cycles. Trees are connected acyclic undirected graphs.

Directed acyclic graphs are called DAGs. They arise naturally in scheduling problems, where a directed edge \((x, y)\) indicates that \( x \) must occur before \( y \).

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**Implicit vs. Explicit Graphs**

Many graphs are not explicitly constructed and then traversed, but built as we use them.

A good example arises in backtrack search.
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**Embedded vs. Topological Graphs**

A graph is *embedded* if the vertices and edges have been assigned geometric positions.

Example: TSP or Shortest path on points in the plane.
Example: Grid graphs.
Example: Planar graphs.

**Labeled vs. Unlabeled Graphs**

In *labeled* graphs, each vertex is assigned a unique name or identifier to distinguish it from all other vertices.

An important graph problem is *isomorphism testing*, determining whether the topological structure of two graphs are in fact identical if we ignore any labels.

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**The Friendship Graph**

Consider a graph where the vertices are people, and there is an edge between two people if and only if they are friends.

This graph is well-defined on any set of people: SUNY SB, New York, or the world.
What questions might we ask about the friendship graph?

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**If I am your friend, does that mean you are my friend?**

A graph is *undirected* if \((x, y)\) implies \((y, x)\). Otherwise the graph is directed.
The “heard-of” graph is directed since countless famous people have never heard of me!
The “had-sex-with” graph is presumably undirected, since it requires a partner.

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**Visualization and human computation: “brain exercise”**

**Perception of and experience**

- Simple example of 5 entities (persons) and their relationships
- Who would you prefer to be?
  - Who wouldn’t you want to be?
- And what if the relationship means “company A sells to company B”?
- What if relationship means “love”?

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**Am I my own friend?**

An edge of the form \((x, x)\) is said to be a *loop*.
If \(x\) is \(y\)’s friend several times over, that could be modeled using *multi-edges*, multiple edges between the same pair of vertices.
A graph is said to be *simple* if it contains no loops and multiple edges.
Am I linked by some chain of friends to the President?

A path is a sequence of edges connecting two vertices. Since Mel Brooks is my father’s-sister’s-husband’s cousin, there is a path between me and him.

How close is my link to the President?

If I were trying to impress you with how tight I am with Mel Brooks, I would be much better off saying that Uncle Lenny knows him than to go into the details of how connected I am to Uncle Lenny. Thus we are often interested in the shortest path between two nodes.

Is there a path of friends between any two people?

A graph is connected if there is a path between any two vertices. A directed graph is strongly connected if there is a directed path between any two vertices.

Who has the most friends?

The degree of a vertex is the number of edges adjacent to it.

Line graph

Given a graph $G$, its line graph $L(G)$ is a graph such that

- each vertex of $L(G)$ represents an edge of $G$; and
- two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint ("are adjacent") in $G"
Complete graph

- Every node is connected to every other node

- Clique – fully connected subgraph of a graph

Subgraph

- Subset of vertices (m) \( V' \) is a subset of \( V \)
- Subset of edges (n) \( E' \) is a subset of \( E \), s.t. \( \{u,v\} \) in \( E' \) if \( u,v \) both in \( V' \)

- Nr of different possible graphs of size \( m,n \) is huge

How many different subgraphs does a complete graph have?

- How many?
  - \( 2^m \) different subsets of vertices (\( = m! \))
  - Likewise, nr of edges is any subset of set of edges...

- 5 nodes \( \Rightarrow \) \( 5^2/2 \) different possible undirected edges without self-loops

- Calculate the possibility of each edge being present or not...

- Directed: \( \rightarrow, <-, <-> \), none (4 options)

Representation of Graphs

- Adjacency list: \( \Theta(V+E) \)
  - Preferred for sparse graph
  - \( |E| < |V|^2 \)
  - \( \text{Adj}[u] \) contains all the vertices \( v \) such that there is an edge \( (u,v) \) \( \in \) \( E \)
  - Weighted graph: \( w(u,v) \) is stored with vertex \( v \) in \( \text{Adj}[u] \)
  - No quick way to determine if a given edge is present in the graph

- Adjacency matrix: \( \Theta(V^2) \)
  - Preferred for dense graph
  - Symmetry for undirected graph
  - Weighted graph: store \( w(u,v) \) in the \( (u,v) \) entry
  - Easy to determine if a given edge is present in the graph

Representation For An Undirected Graph

![Undirected Graph Diagram](image)

Figure 22.1: Two representations of an undirected graph. (a) An undirected graph \( G \) having five vertices and seven edges. (b) An adjacency list representation of \( G \). (c) The adjacency-matrix representation of \( G \).

Representation For A Directed Graph

![Directed Graph Diagram](image)

Figure 22.2: Two representations of a directed graph. (a) A directed graph \( G \) having six vertices and eight edges. (b) An adjacency-list representation of \( G \). (c) The adjacency-matrix representation of \( G \).
Breadth-First Search (BFS)

- Graph search: given a source vertex \( s \), explores the edges of \( G \) to discover every vertex that is reachable from \( s \).
  - Compute the distance (smallest number of edges) from \( s \) to each reachable vertex.
  - Produce a breadth-first tree with root \( s \) that contains all reachable vertices.
  - Compute the shortest path from \( s \) to each reachable vertex.
  - BFS discovers all vertices at distance \( k \) from \( s \) before discovering any vertices at distance \( k+1 \).

Data Structure for BFS

- Adjacency list
- color[u] for each vertex
  - WHITE if \( u \) has not been discovered
  - BLACK if \( u \) and all its adjacent vertices have been discovered
  - GRAY if \( u \) has been discovered, but has some adjacent white vertices
- Frontier between discovered and undiscovered vertices
- \( d[u] \) for the distance from [source] \( s \) to \( u \)
- \( \pi[u] \) for predecessor of \( u \)
- FIFO queue \( Q \) to manage the set of gray vertices
  - \( Q \) stores all the gray vertices

Simple BFS from \( n \)

```
enqueue( Q, n )

while Q not empty
  u = dequeue( Q)
  process u
  for each v in Adjacency( u )  // discover neighbours
    if v not yet discovered
      then enqueue( Q, v )
```
Initialization

Set up $s$ and initialize $Q$

$\text{BFS}(G, s)$

1. for each vertex $u \in V[G] - \{s\}$
2. \hspace{1em} do $color[u] \leftarrow$ WHITE
3. \hspace{2em} $d[u] \leftarrow \infty$
4. \hspace{3em} $\pi[u] \leftarrow$ NIL
5. \hspace{4em} $color[s] \leftarrow$ GRAY
6. \hspace{5em} $d[x] \leftarrow 0$
7. \hspace{6em} $\pi[x] \leftarrow$ NIL
8. \hspace{7em} $Q \leftarrow \emptyset$
9. \hspace{8em} ENQUEUE($Q, s$)
10. while $Q \neq \emptyset$

\begin{align*}
\text{Explore all the vertices adjacent to } & u \text{ and update } d, \pi \text{ and } Q \\
\text{do } u \leftarrow \text{DEQUEUE}(Q) \\
\text{for each } v \in \text{Adj}[u] \text{ do if } & color[v] = \text{WHITE} \\
\text{then color[v] } & \leftarrow \text{GRAY} \\
\text{d[v] } & \leftarrow d[u] + 1 \\
\pi[v] & \leftarrow u \\
\text{ENQUEUE}(Q, v) \\
\text{color[u] } & \leftarrow \text{BLACK}
\end{align*}

Example (BFS)

(Courtesy of Prof. Jim Anderson)

Example (BFS)

Example (BFS)

Example (BFS)
Example (BFS)

1. Q: x y u
   2. 2 2 3

Example (BFS)

2. Q: v u y
   3. 2 3 3

Example (BFS)

3. Q: u y
   4. 3 3

Example (BFS)

4. Q: ∅
Analysis of BFS

- $O(|V|+|E|) = O(n+m)$
  - Each vertex is enqueued (O(1)) at most once $\Rightarrow$ O(n)
    - No vertex is repainted white
  - Each adjacency list is scanned at most once $\Rightarrow$ O(m)

Shortest path

- Print out the vertices on a shortest path from $s$ to $v$

```
PRINT-PATH(G, s, v)
1 if $v = s$
2 then print $s$
3 else if $\pi[v] = NIL$
4 then print “no path from” $s$ “to” $v” “exists”
5 else PRINT-PATH(G, s, $\pi[v]$)
6 print $v$
```

PRINT-PATH Illustration

```
PRINT-PATH(G, s, u)
print u
```

```
PRINT-PATH(G, s, t)
print t
```

```
PRINT-PATH(G, s, w)
print w
```

```
PRINT-PATH(G, s, s)
output: s w t u
```

Connected Components

The connected components of an undirected graph are the separate “pieces” of the graph such that there is no connection between the pieces.

Many seemingly complicated problems reduce to finding or counting connected components. For example, testing whether a puzzle such as Rubik’s cube or the 15-puzzle can be solved from any position is really asking whether the graph of legal configurations is connected.

Anything we discover during a BFS must be part of the same connected component. We then repeat the search from any undiscovered vertex (if one exists) to define the next component, until all vertices have been found.

Two-Coloring Graphs

The vertex coloring problem seeks to assign a label (or color) to each vertex of a graph such that no edge links any two vertices of the same color.

A graph is bipartite if it can be colored without conflicts while using only two colors. Bipartite graphs are important because they arise naturally in many applications.

For example, consider the “has-sex-with” graph in a heterosexual world. Men have sex only with women, and vice versa. Thus gender defines a legal two-coloring.
### Bipartite graphs

- people and groups
- men-women
- Stable marriage
  - find matching that will not be "broken" by inevitable divorces
- Apples and Oranges

### Finding a Two-Coloring

We can augment breadth-first search so that whenever we discover a new vertex, we color it the opposite of its parent.

```c
int main()
{
    int G[100][100];
    int n = 100;
    int m = 100;
    for (int i = 0; i < n; i++)
        for (int j = 0; j < m; j++)
            G[i][j] = 0;
    for (int i = 0; i < n; i++)
        for (int j = 0; j < m; j++)
            G[i][j] = (i + j) % 2;

    int color[n];
    for (int i = 0; i < n; i++)
        color[i] = 0;
    color[0] = 1;
    int queue[n];
    int front = 0;
    int rear = 0;
    queue[rear] = 0;
    rear = (rear + 1) % n;
    while (front != rear)
    {
        int u = queue[front];
        front = (front + 1) % n;
        int v;
        for (v = 0; v < m; v++)
            if (G[u][v] == 1)
                break;
        if (v == m)
            continue;
        if (color[v] == 0)
        {
            color[v] = !color[u];
            queue[rear] = v;
            rear = (rear + 1) % n;
        }
    }
    for (int i = 0; i < n; i++)
        printf("%d ", color[i]);
    printf("\n");
    return 0;
}
```

### Problem of the Day

Prove that in a breadth-first search on a undirected graph $G$, every edge in $G$ is either a tree edge or a cross edge, where a cross edge $(x, y)$ is an edge where $x$ is neither an ancestor or descendent of $y$.

### Depth-First Search (DFS)

- DFS: search deeper in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex $v$ that still has unexplored edges leaving it
  - When all of $v$’s edges have been explored (finished), the search backtracks to explore edges leaving the vertex from which $v$ was discovered
  - This process continues until we have discovered all the vertices that are reachable from the original source vertex
  - If any undiscovered vertices remain, then one of them is selected as a new source and the search is repeated from that source
  - The entire process is repeated until all vertices are discovered
- DFS will create a forest of DFS-trees

### Simple DFS from n

```c
int main()
{
    int n;
    scanf("%d", &n);
    int G[n][n];
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            scanf("%d", &G[i][j]);

    int color[n];
    for (int i = 0; i < n; i++)
        color[i] = 0;

    int stack[n];
    int top = 0;
    stack[top] = 0;
    while (top > 0)
    {
        int u = stack[top-1];
        top--;
        int v;
        for (v = 0; v < n; v++)
            if (G[u][v] == 1)
                break;
        if (v == n)
            continue;
        if (color[v] == 0)
        {
            color[v] = 1;
            stack[top] = v;
            top++;
        }
    }
    for (int i = 0; i < n; i++)
        printf("%d ", color[i]);
    printf("\n");
    return 0;
}
```
Recursive:

DFS( u ) :
  process u
  for each v in Adjacency( u )
    DFS( v ) if v undiscovered

Data Structure for DFS

- Adjacency list
- color[u] for each vertex
  - WHITE if u has not been discovered
  - GRAY if u is discovered but not finished
  - BLACK if u is finished
- Timestamps: 1 ≤ d[u] < f[u] ≤ 2|V|
  - d[u] records when u is first discovered (and grayed)
  - f[u] records when the search finishes examining u’s adjacency list (and blacken u)
- π[u] for predecessor of u

The Key Idea with DFS

A depth-first search of a graph organizes the edges of the graph in a precise way. In a DFS of an undirected graph, we assign a direction to each edge, from the vertex which discover it:

DFS: initialise and visit all yet unexplored vertices

DFS(G) :
1. for each vertex u ∈ V[G]
2.   do color[u] ← WHITE
3.     π[u] ← NIL
4.     time ← 0
5.   for each vertex u ∈ V[G]
6.     do if color[u] = WHITE
7.       then DFS-visit( u )

DFS-visit – visit all reachable nodes

DFS-Visit( u )
1. color[u] ← GRAY  // White vertex u has just been discovered.
2. time ← time +1
3. d[u] ← time
4. for each v ∈ Adj[u]  // Explore edge (u, v).
5.     do if color[v] = WHITE
6.       then π[v] ← u
7.     DFS-Visit( v )
8. color[u] ← BLACK  // Blacken u; it is finished.
9. f[u] ← time ← time +1
Properties of DFS

- Time complexity: $\Theta(V+E)$
  - Loops on lines 1–3 and 5–7 of DFS: $\Theta(V)$
    - DFS-VISIT
      - Called exactly once for each vertex
      - Loops on lines 4–7 for a vertex $v$: $|\text{Adj}[v]|$
      - Total time $= \sum \limits_{v \in V} |\text{Adj}[v]| = \Theta(E)$

- DFS results in a forest of trees
- Discovery and finishing times have parenthesis structure
Another Example of DFS

DFS
- Stack

BFS
- Queue

Randomised search
Use:
Randomized Queue
Parenthesis Theorem

Theorem 22.7
For all $u, v$, exactly one of the following holds:
1. $d(u) < f(u) < d(v) < f(v)$ or $d(v) < f(v) < d(u) < f(u)$ and neither $u$ nor $v$ is a descendant of the other.
2. $d(u) < f(v) < f(u)$ and $v$ is a descendant of $u$.
3. $d(v) < f(u) < f(v)$ and $u$ is a descendant of $v$.

- So $d(x) < f(x) < f(y)$ cannot happen.
- Like parentheses:
  - OK: $( ) [ ] ( [ ] ) [ ( ) ]$
  - Not OK: $( [ ) ] [ ( ] )$

Corollary
$v$ is a proper descendant of $u$ if and only if $d(v) < d(u) < f(v) < f(u)$.

Example (Parenthesis Theorem)

Proof

We begin with the case in which $d(v) < d(u)$.

- There are two subcases to consider, according to whether $d(u) < f(u)$ or not.
- The first subcase occurs when $d(u) < f(u)$, so $v$ was discovered while $u$ was still gray. This implies that $v$ is a descendant of $u$. Moreover, since $v$ was discovered more recently than $u$, all of its outgoing edges are explored, and $v$ is finished, before the search returns to and finishes $u$. In this case, therefore, the interval $[d(v), f(v)]$ is entirely contained within the interval $[d(u), f(u)]$.
- In the other subcase, $f(u) < d(u)$, and inequality (22.2) implies that the intervals $[d(u), f(u)]$ and $[d(v), f(v)]$ are disjoint.
- Because the intervals are disjoint, neither vertex was discovered while the other was gray, and so neither vertex is a descendant of the other.
- The case in which $d(v) < d(u)$ is similar, with the roles of $u$ and $v$ reversed in the above argument.
Depth-First Trees

- Predecessor subgraph defined slightly different from that of BFS.
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V$ and $\pi[v] \neq \text{NIL}\}$.
  - How does it differ from that of BFS?
  - The predecessor subgraph $G_\pi$ forms a depth-first forest composed of several depth-first trees. The edges in $E_\pi$ are called tree edges.

Definition:
Forest: An acyclic graph $G$ that may be disconnected.

White-path Theorem

**Theorem 22.9**

$v$ is a descendant of $u$ if and only if at time $d[u]$, there is a path $u \rightarrow v$ consisting of only white vertices. (Except for $u$, which was just colored gray.)

Classification of Edges

- **Tree edge**: in the depth-first forest. Found by exploring $(u, v)$. -- $v$ was white
- **Back edge**: $(u, v)$, where $u$ is a descendant of $v$ (in the depth-first tree). -- $v$ was gray
- **Forward edge**: $(u, v)$, where $v$ is a descendant of $u$, but not a tree edge. -- $v$ was black and $d[u] < d[v]$
- **Cross edge**: any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.
  -- $v$ was black and $d[u] > d[v]$

**Theorem:**
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

Types of edges in BFS

- **Tee edge**
- **Back edge**
- **Forward edge**
- **Cross edge**

Edge Classification for DFS

Every edge is either:
1. A Tree Edge
2. A Back Edge
3. A Forward Edge
4. A Cross Edge

On any particular DFS or BFS of a directed or undirected graph, each edge gets classified as one of the above.

**DFS: Tree Edges and Back Edges Only**

The reason DFS is so important is that it defines a very nice ordering to the edges of the graph.
In a DFS of an undirected graph, every edge is either a tree edge or a back edge.

Why? Suppose we have a forward edge. We would have encountered $(4, 1)$ when expanding 4, so this is a back edge.
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No Cross Edges in DFS

Suppose we have a cross-edge

\[ \text{No Cross Edges in DFS} \]

DFS Application: Finding Cycles

Back edges are the key to finding a cycle in an undirected graph. Any back edge going from \( x \) to an ancestor \( y \) creates a cycle with the path in the tree from \( y \) to \( x \).

Lemma – DAG acyclicity

- DAG is acyclic if and only if DFS of \( G \) yields no back edges
  - Suppose that there is a back edge \((u, v)\). Then vertex \( v \) is an ancestor of vertex \( u \) in the depth-first forest. There is thus a path from \( v \) to \( u \) in \( G \), and the back edge \((u, v)\) completes a cycle
  - Suppose that \( G \) contains a cycle \( c \). We show that a DFS of \( G \) yields a back edge. Let \( v \) be the first vertex to be discovered in \( c \), and let \((u, v)\) be the preceding edge in \( c \). At time \( \delta(v) \), the vertices of \( c \) form a path of white vertices from \( v \) to \( u \). By the white-path theorem (Theorem 22.9), vertex \( u \) becomes a descendant of \( v \) in the depth-first forest. Therefore, \((u, v)\) is a back edge.

Articulation Vertices

Suppose you are a terrorist, seeking to disrupt the telephone network. Which station do you blow up?

An articulation vertex is a vertex of a connected graph whose deletion disconnects the graph. Clearly connectivity is an important concern in the design of any network. Articulation vertices can be found in \( O((n+m)) \) – just delete each vertex to do a DFS on the remaining graph to see if it is connected.

A Faster \( O(n + m) \) DFS Algorithm

In a DFS tree, a vertex \( v \) (other than the root) is an articulation vertex if \( v \) is not a leaf and some subtree of \( v \) has no back edge incident until a proper ancestor of \( v \).

Problems 22-2: Articulation points, bridges, and biconnected components

Let \( (V, E) \) be a connected, undirected graph. An articulation point of \( G \) is a vertex whose removal disconnects \( G \). A bridge of \( G \) is an edge whose removal disconnects \( G \). A biconnected component of \( G \) is a maximal set of edges such that any two edges in the set lie on a common simple cycle. Figure 22.12 illustrates these definitions. We can determine articulation points, bridges, and biconnected components using depth-first search. Let \( G_u = (V, E_u) \) be a depth-first tree of \( G \).

Figure 22.12: The articulation points, bridges, and biconnected components of a connected, undirected graph for use in Problem 22.2. The articulation points are the heavily shaded vertices, the bridges are the heavily shaded edges, and the biconnected components are the edges in the shaded regions, with a box numbering shown.

A. Prove that the root of \( G_u \) is an articulation point of \( G \) if and only if it has at least two children in \( G_u \).
B. Let \( v \) be a non-root vertex of \( G_u \). Prove that \( v \) is an articulation point of \( G \) if and only if \( v \) has a child such that there is no back edge from \( v \) to any descendant of \( v \) to a proper ancestor of \( v \).

...
TOPOLOGICAL SORTING

Topological Sort

• A topological sort of a directed acyclic graph (DAG) is a linear order of all its vertices such that if G contains an edge (u, v), then u appears before v in the ordering
  – if the graph contains cycles, no linear ordering is possible.
  – A topological sort can be viewed as an ordering of its vertices along a horizontal line so that all directed edges go from left to right
• DAG are used in many applications to indicate precedence among events

Topological Sorting

A directed, acyclic graph has no directed cycles.

A topological sort of a graph is an ordering on the vertices so that all edges go from left to right.
DAGs (and only DAGs) has at least one topological sort (here G, A, B, C, F, E, D).

Ordering?

Topological Sort

• $\Theta(V+E)$

**TOPOLOGICAL-SORT(G)**

1. call DFS(G) to compute finishing times $f[v]$ for each vertex $v$
2. as each vertex is finished, insert it onto the front of a linked list
3. return the linked list of vertices
Example

Linked List:

Example

Linked List:

Example

Linked List:

Example

Linked List:
Theorem: Correctness of Topological sort

- **TOPOLOGICAL-SORT(G)** produces a topological sort of a directed acyclic graph G
  - Suppose that DFS is run on a given DAG G to determine finishing times for its vertices. It suffices to show that for any pair of distinct vertices u, v, if there is an edge in G from u to v, then \( f(v) < f(u) \).
    - The linear ordering is corresponding to finishing time ordering
  - Consider any edge \((u, v)\) explored by DFS(G). When this edge is explored, v cannot be gray (otherwise, \((u, v)\) will be a back edge). Therefore v must be either white or black
    - If v is white, v becomes a descendant of u, \( f(v) < f(u) \) (ex. pants & shoes)
    - If v is black, it has already been finished, so that \( f(v) \) has already been set \( f(v) < f(u) \) (ex. belt & jacket)
STRONGLY CONNECTED COMPONENTS

Strongly Connected Components
- \( G \) is strongly connected if every pair \((u, v)\) of vertices in \( G \) is reachable from one another.
- A **strongly connected component (SCC)** of \( G \) is a maximal set of vertices \( C \subseteq V \) such that for all \( u, v \in C \), both \( u \rightarrow v \) and \( v \rightarrow u \) exist.

\[ G_{SCC} = (V_{SCC}, E_{SCC}) \]
- \( V_{SCC} \) has one vertex for each SCC in \( G \).
- \( E_{SCC} \) has an edge if there’s an edge between the corresponding SCC’s in \( G \).

\( G_{SCC} \) for the example considered:

**Proof:**
- Suppose there is a path \( v' \sim v \) in \( G \).
- Then there are paths \( u \sim u' \sim v' \) and \( v \sim v' \sim u \) in \( G \).
- Therefore, \( u \) and \( v' \) are reachable from each other, so they are not in separate SCC’s.

Component Graph
- \( G^{SCC} = (V^{SCC}, E^{SCC}) \).
- \( V^{SCC} \) has one vertex for each SCC in \( G \).
- \( E^{SCC} \) has an edge if there’s an edge between the corresponding SCC’s in \( G \).
- \( G^{SCC} \) for the example considered:

Transpose of a Directed Graph
- \( G^T = \text{transpose} \) of directed \( G \).
  - \( G^T = (V, E^T), E^T = \{(u, v) : (v, u) \in E\} \).
  - \( G^T \) is \( G \) with all edges reversed.
- Can create \( G^T \) in \( O(V + E) \) time if using adjacency lists.
- \( G \) and \( G^T \) have the same SCC’s. \((u \text{ and } v \text{ are reachable from each other in } G \text{ if and only if reachable from each other in } G^T.\)
Algorithm to determine SCCs

**SCC**

1. call DFS(G) to compute finishing times \( f[u] \) for all \( u \)
2. compute \( G^T \)
3. call DFS(\( G^T \)), but in the main loop, consider vertices in order of decreasing \( f[u] \) (as computed in first DFS)
4. output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

**Time:** \( \Theta(V + E) \).

**Example:** On board.

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**Example**

(Courtesy of Prof. Jim Anderson)

**Example**

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**How does it work?**

- **Idea:**
  - By considering vertices in second DFS in decreasing order of finishing times from first DFS, we are visiting vertices of the component graph in topologically sorted order.
  - Because we are running DFS on \( G^T \), we will not be visiting any \( v \) from a \( u \), where \( v \) and \( u \) are in different components.

- **Notation:**
  - \( d[u] \) and \( f[u] \) always refer to first DFS.
  - Extend notation for \( d \) and \( f \) to sets of vertices \( U \subseteq V \):
    - \( d(U) = \min_{u \in U} d[u] \) (earliest discovery time)
    - \( f(U) = \max_{u \in U} f[u] \) (latest finishing time)

**SCCs and DFS finishing times**

**Lemma 22.14**

Let \( C \) and \( C' \) be distinct SCC’s in \( G = (V, E) \). Suppose there is an edge \( (u, v) \in E \) such that \( u \in C \) and \( v \in C' \). Then \( f(C) > f(C') \).

**Proof:**

- Case 1: \( d(C) < d(C') \)
  - Let \( x \) be the first vertex discovered in \( C \).
  - At time \( d(x) \), all vertices in \( C \) are white. Thus, there exist paths of white vertices from \( x \) to all vertices in \( C \) and \( C' \).
  - By the white-path theorem, all vertices in \( C' \) are descendants of \( x \) in depth-first tree.
  - By the parenthesis theorem, \( f(x) = f(C) > f(C') \).
SCCs and DFS finishing times

Lemma 22.14
Let \( C \) and \( C' \) be distinct SCC’s in \( G = (V, E) \). Suppose there is an edge \((u, v) \in E\) such that \( u \in C \) and \( v \in C' \). Then \( f(C) > f(C') \).

Proof:
1. Case 2: \( d(C) > d(C') \)
   - Let \( y \) be the first vertex discovered in \( C' \).
   - At time \( d[y] \), all vertices in \( C' \) are white and there is a white path from \( y \) to each vertex in \( C' \) = \{ all vertices in \( C' \) ≤ \( C \) or \( C \) doesn’t have a path to \( y \). \}
   - Therefore, \( y \) is also white.
   - By earlier lemma, since there is an edge \((u, v) \in E\) \( u \in C \) and \( v \in C' \), we cannot have a path from \( C' \) to \( C \).
   - Therefore, \( y \) is reachable from \( x \).
   - Therefore, for all \( w \in C \) \( f(C) > f(C') \), which implies that \( f(C) > f(C') \).

SCCs and DFS finishing times

Corollary 22.15
Let \( C \) and \( C' \) be distinct SCC’s in \( G = (V, E) \). Suppose there is an edge \((u, v) \in E\) such that \( u \in C \) and \( v \in C' \). Then \( f(C) < f(C') \).

Proof:
- \( (u, v) \in E \Rightarrow (v, u) \in E \).
- Since SCC’s of \( G \) and \( G^T \) are the same, \( f(C) < f(C') \), by Lemma 22.14.

Correctness of SCC

- When we do the second DFS, on \( G^T \), start with SCC \( C \) such that \( f(C) \) is maximum.
  - The second DFS starts from some \( x \in C \), and it visits all vertices in \( C \).
  - Corollary 22.15 says that since \( f(C) > f(C') \) for all \( C \neq C' \), there are no edges from \( C' \) to \( C \).
  - Therefore, DFS will visit only vertices in \( C \).
  - Which means that the depth-first tree rooted at \( x \) contains exactly the vertices of \( C \).

Why does strongly connected component method work?

- Seel CLRS (2-3 pages)