Math and Music

Symmetry in Music

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Symmetry

Definition

A *symmetry* is a set of *transformations* applied to a *structure*, such that the transformations *preserve* the properties of the structure.

Analysis

This does not say us much:

- What structures? Do all structures fit?
- What transformations? Are there some constraints?
- What is preservation? What properties need to be preserved?

Is eating an ice-cream a preserving tranformation? Does it impose symmetry on ice-creams?

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There are however some sane requirements:

- Preserving transformations composition should be preserving
- Preserving transformations should be invertable, there inverse should also be preserving

Welcome to the group theory!

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Group Theory

Definition

Group $(G, \cdot)$ is a set G and a binary operation $\cdot$ that satisfy:

1. For any $g, h, k \in G$ we have $g \cdot (h \cdot k) = (g \cdot h) \cdot k$
2. There exists $e \in G$ so that for any $g \in G$ we have $g \cdot e = e \cdot g = g$
3. For each $g \in G$ there exists $g^{-1} \in G$ so that $g \cdot g^{-1} = g^{-1} \cdot g = e$

Note that commutative law ($g \cdot h = h \cdot g$) is not required.

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Permutations

Definition

For a set X *permutation* is any bijective function of type $f : X \to X$.

Example

If $X = \{1, 2, 3, 4, 5\}$ then one possible permutation is:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

It can also be written in cycle notation as $f = (1, 3, 4)(2, 5)$.

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Permutations

Definition

The set of all permutations of X form a *symmetric* group $\mathrm{Symm}(X)$:

- Identity function gives e
- Function composition is the binary operation and is associative
- Every bijective function has an inverse

For $|X| = n$ we have $|\mathrm{Symm}(X)| = n!$.

Definition

- A subgroup H is a subset of group G that includes e and is closed under inherited composition and inverse.
- A subgroup of $\mathrm{Symm}(X)$ is called a *permutation group*

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Change Ringing

Description

- Large bells cannot be rang at will
- Bells are rang by series, where each bell is rang at least once
- Therefore each bell can change only ring one position before or later in the next series
- The goal is to get all permutations this way (why is it possible?)

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Cayley's Theorem

Definition

If G and H are groups the a *homomorphism* $f : G \to H$ is a function that satisfies:

$$f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$$

Injective homomorphism is called *monomorphism*, surjective *epimorphism*, bijective *isomorphism*.
Note that if f is a monomorphism, then G is isomorphic to its image $f[G]$, which is a subgroup of H.

Cayley's Theorem

Theorem

Let G be a group and let f be a function from G to $\mathrm{Symm}(G)$ defined by $f(g)(h) = g \cdot h$. Then f is a monomorphism and so G is isomorphic with a subgroup of $\mathrm{Symm}(G)$.

Proof

$$f(g^{-1})(f(g)(h)) = f(g^{-1})(g \cdot h) = g^{-1} \cdot g \cdot h = h$$

$$f(g_1 \cdot g_2)(h) = g_1 \cdot g_2 \cdot h = f(g_1)(g_2 \cdot h) = (f(g_1) \cdot f(g_2))(h)$$

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Pitch arithmetic

Definition

We can form an obvious $\mathbb{Z}_{12}$ group with addition by lifting each pitch to a number:

C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B
0	1	2	3	4	5	6	7	8	9	10	11

The addition in this group works by the usual congruence laws:

$$6 + 8 \equiv 2 \mod 12$$

Using Cayley's theorem we can lift each element i to a corresponding permutation T_i that adds i to the input. We can then extend T_i in an obvious way to sequences of fixed length n by applying it to each of the elements. This will also be a group.

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Cyclic groups

Definition

An element g in *cyclic* group G is said to *generate* G if every $g' \in G$ is represented by $g' = g^k$ for some k (identity element is considered to be $e = g^0$). Both $\mathbb{Z}$ and $\mathbb{Z}_n$ are cyclic. The group of $\{T_i\}$ is also cyclic, so we can pick an element T as the generator and write $T_i = T^i$.

Pitch arithmetic

Chopin's *Etùde*, Op. 25 No. 10:

Pitch arithmetic

Definition

Chopin's *Etùde*, Op. 25 No. 10 first two bars consist of the pitches:

6–5–6 7–8–9 8–7–8 9–10–11

10–9–10 11–0–1 0–11–0 1–2–3

Taking x for sequence 6 5 6 7 8 9 we can rewrite it as:

$$x\ T_2(x) \mid T_4(x)\ T_6(x)$$

If last sequence (1–2–3) is taken as y, we can rewrite the next two bars as:

$$T_1(y)\ T_2(y)\ T_3(y)\ T_4(y) \mid T_5(y)\ T_6(y)\ T_6(y)\ T_7(y)$$

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Sequence Operations

Definition

The inversion operation ($I(g) = g^{-1}$). It can also be extended on sequences, e.g. taking $\mathrm{x} = 3\ 0\ 8$:

$$I(\mathrm{x}) = 9\ 0\ 4$$

Retrograde operation (R) can only be defined on sequences and reverses the sequence order:

$$R(\mathrm{x}) = 8\ 0\ 3$$

We have the following relations among T_i, I and R:

$$T_{12} = e \quad T_n \cdot R = R \cdot T_n \quad T_n \cdot I = I \cdot T_{-n} \quad R \cdot I = I \cdot R$$

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Cartesian Products

Definition

R and *id* with composisiton form a group G isomorphic to $\mathbb{Z}_2$. T_i-s form a group H isomorphic to $\mathbb{Z}_{12}$. We can lift them to a cartesian product $G \times H$. Since we know that R and T_i commute, we can also form an *inner direct product* given by an isomorphism $f : G \times H \to K$ where

$$f(g, h) = g \cdot h$$

This will form a single group with elements of sort T_i and $R \cdot T_i$.

Dihedral Groups

Definition

T_i and I do not commute, but form a *dihedral* group.
A dihedral group has two elements g and h so that $h^2 = e$ and $g \cdot h = h \cdot g^{-1}$. Every element is thus either in form g^i or $g^i \cdot h$. The group has $2n$ elements (where n is order of g) and is written D_{2n}. Operations T_i and $T_i \cdot I$ form a dihedral group D_{24}. All operations together:

$$T_i \quad T_i \cdot R \quad T_i \cdot I \quad T_i \cdot R \cdot I$$

form an inner direct product group that is isomorphic to $D_{24} \times \mathbb{Z}_2$ that has 48 elements.

Twelve Tone Rows

Description

- Twelve tone row approach ensures that all pitches appear uniformly the music piece
- A base row is chosen that has all twelve different pitches
- The music piece is composed applying T, R and I to that row
- Of course we also have time as a player, so we can overlap and stretch the rows
- There are 9985920 variants of base tone row that are not equivalent under T, R or I and their compositions.

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