Mathematics of Sound and Music

Seminar 3: Fourier Theory

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Outline

1. Decomposition of functions into sinewaves

2. Fourier series

3. Fourier transform

4. Why Fourier transform?

Why sinewaves?

- **Physics and physiology:** We generate and perceive sound as a collection of sinewaves.
- **Mathematics:** A sinewave is the *eigensignal* of a *linear time-invariant transformation*.

Decomposition: How's that?

- For example: $f(t) = 2\sin(t) + 3\sin(2t) - \sin(1.3t)$.
- Compare to: $\mathbf{v} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \cdots + \alpha_n \mathbf{b}_n$.
- If $\{\mathbf{b}_i\}$ is an *orthonormal basis*, then

$$\alpha_i = \langle \mathbf{v}, \mathbf{b}_i \rangle.$$

Decomposition: How's that?

As we're dealing with an *infinite-dimensional space* we'll be decomposing f as a *series*:

$$f(t) = \sum_{i=0}^{\infty} \alpha_i b_i(t),$$

or an *integral*:

$$f(t) = \int_{w=0}^{\infty} \alpha(w) b_w(t) \mathrm{d}w.$$

Compare to Taylor series:

$$f(x) = \sum_{i=0}^{n} \alpha_n x^n.$$

Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt)).$$

The family of functions 1, $\sin(nt)$, $\cos(nt)$ is *orthogonal*:

$$\langle \cos(mt), \sin(nt) \rangle = \int_0^{2\pi} \cos(mt) \sin(nt) \mathrm{d}t = 0$$

$$\oint \left(x^2 + y^2\right) \mathrm{d}\Omega$$

Fourier series

$$\langle \cos(mt), \cos(nt) \rangle = \int_0^{2\pi} \cos(mt)\cos(nt)\mathrm{d}t = \begin{cases} 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \sin(mt), \sin(nt) \rangle = \int_0^{2\pi} \sin(mt)\sin(nt)\mathrm{d}t = \begin{cases} \pi & \text{if } m = n > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\oint \left(x^2 + y^2\right) \mathrm{d}\Omega$$

Fourier series

Hence,

$$a_m = \frac{1}{\pi}\langle f(t), \cos(mt)\rangle = \int_0^{2\pi} f(t)\cos(mt)\mathrm{d}t,$$

$$b_m = \frac{1}{\pi}\langle f(t), \sin(mt)\rangle = \int_0^{2\pi} f(t)\sin(mt)\mathrm{d}t.$$

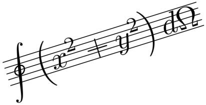
Complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{int},$$

Fourier transform

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t} \mathrm{d}\nu,$$

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t} \mathrm{d}t.$$



Fourier transform properties

$$\alpha f(t) + g(t) \sim \alpha F(\nu) + G(\nu) \tag{1}$$

$$F(t) \sim f(-\nu) \tag{2}$$

$$f(at) \sim \frac{1}{|a|} F\left(\frac{\nu}{a}\right) \tag{3}$$

$$f(t - t_0) \sim e^{-2\pi i \nu t_0} F(\nu) \tag{4}$$

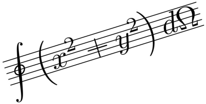
Fourier transform properties

$$f(t)e^{2\pi i\nu_0 t} \sim F(\nu - \nu_0) \tag{5}$$

$$f(t) * g(t) \sim F(\nu)G(\nu) \tag{6}$$

$$f'(t) \sim 2\pi\nu F(\nu) \tag{7}$$

$$\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |F(\nu)|^2 \mathrm{d}nu \tag{8}$$



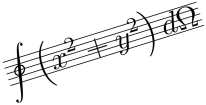
Windowed Fourier transform

$$\mathcal{F}(t_0, \nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t} \psi(t - t_0) \mathrm{d}t.$$

Discrete Fourier transform

$$F_n = \sum_{k=0}^{N-1} f_k e^{-2\pi i n k/N}$$

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{-2\pi i n k/N}$$



Why Fourier transform?

- Detection of periodicity in biological, financial, meteorological time series
- Solutions of differential equations
- Image processing and compression
- Sound synthesis and analysis
- All kinds of pattern analysis tasks
- ..and more

$$\oint \left(x^2 + y^2\right) d\Omega$$

Questions?

$$\oint \left(x^2 + y^2\right) d\Omega$$