Trees

- Some of the very basic essence of computer science and programming
- Chapter 5 – “The Tree Data Model” (pp 223-285) in
  - Foundations of Computer Science: C Edition
  - Alfred V. Aho, Jeffrey D. Ullman
  - W. H. Freeman (October 15, 1994)

Tree

- Acyclic graph
  - root of a tree
  - children, parents, siblings, internal nodes, leaves
- Binary tree – node has 0, 1, or 2 children

Data model

- Abstraction
- File directory system
- Hierarchical organisation structure
  - divide and conquer
- Hierarchical controlled vocabulary (simple ontology)
- Syntactic structure of a (sentence in a) language
- Syntax – e.g. paired parentheses
- ...

Example: XHTML and CSS

- This defines a single tree
Example: XHTML and CSS

• The nested tags define sub-trees

```html
<html>
<head>
  <title>Hello World!</title>
</head>
<body>
  <h1>This is a <u>Heading</u></h1>
  <p>This is a paragraph with some <u>underlined</u> text.</p>
</body>
</html>
```

• This may be rendered by a web browser

Terminology

- root
- leaf: {D,F,G,J,K,L,M}
- degree B = 2
- degree I = 3
- Every node has 1 parent except root has 0 parents
- Depth = 3
- Level C,E,I,M = 2
- width = 6 (at level 3)
- successor = children
- siblings = {F,G}, {J,K,L}
- Path to K = A,H,I,K

Terminology

- List is also a tree:

Terminology

- Descendants (of B) = B,C,D,E,F,G
- = a subtree with a root B
- Ancestors of I = I,H,A

Every node is connected via a path to root

Terminology

Topologically equal to previous slide

Depends on application if order is important or not
Trie
For P={he, she, his, hers}

Binary trees
- This peach tree is not a binary tree...

Binary Trees
Definition: Any node can have 0, 1 or 2 children
- A full node is a node where both the left and right sub-trees are non-empty trees

Legend:
- full nodes
- neither
- leaf nodes

Basic node structure
Binary Trees

• An empty node or a null sub-tree is any location where a new leaf node could be inserted.

Binary Trees Definition

• A full binary tree is where each node is:
  – a full node, or
  – a leaf node

• This has applications in
  – expression trees and Huffman encoding

Perfect Binary Trees Definition

• Standard definition:
  – A perfect binary tree of height \( h \) is a binary tree where
    • All leaves have the same depth \( h \)
    • All other nodes are full

Perfect Binary Trees Examples

• Perfect binary trees of height \( h = 0, 1, 2, 3 \) and 4

Perfect Binary Trees \( 2^h + 1 - 1 \) Nodes

• Using the recursive definition, both sub-trees are perfect trees of height \( h - k - 1 \)
• By assumption, each sub-tree has \( 2^{k-1} - 1 \) nodes
• Therefore the total number of nodes is
  \[
  (2^h - 1) + 1 + (2^h - 1) = 2^{h+1} - 1
  \]

Complete Binary Trees Definition

• A complete binary tree filled at each depth from left to right:
Complete Binary Trees
Array Storage

- Fill the array following a breadth-first traversal:

left(i) = i*2
right(i) = i*2+1

1 2 3 4 5 6 7 8 9 10 11 12

- To insert another node while maintaining the complete-binary-tree structure, we must insert into the next array location.

Complete Binary Trees
Array Storage

Traversal of a binary tree

Tree-Walk( x )
if x ≠ NULL then
    // pre-order operations
    Tree-Walk( left(x) )
    // in-order operations
    Tree-Walk( right(x) )
    // post-order operations

Traversal of a binary tree - size

int Tree-Size( x )
if x == NULL then return 0
return
    Tree-Size( left(x) ) +
    Tree-Size( right(x) ) + 1

Traversal of a binary tree – parenthesisation

Tree-Walk( x )
if x ≠ NULL then
    print “(”, x.value ;
    Tree-Walk( left(x) ) ;
    Tree-Walk( right(x) ) ;
    print “)” ;
Parenthesisation

( 3 ( 9 ( 14 (17) (15) ) (10 (13) (23) ) ) ) ... 

Binary Trees
Application: Expression Trees

• Expression trees
3(2a + c + a) + b/3 + (a – 2)

• Observations:
  – internal nodes store operators
  – leaf nodes store operands
  – no nodes have just one sub tree
  – the order is not relevant for
    • addition and multiplication (commutative)
    • subtraction and division (non-commutative)
  – to ignore order complete, represent subtraction
    and division as unary operators
    (a/b) = a * b⁻¹
    (a – b) = a + (-b)

Evaluate the expression

```c
int Eval-Tree( x )
int val1, val2;
if x->op == ‘1’ return x->value; // x is a leaf, integer value
else
  val1 = Eval-Tree( x->left )
  val2 = Eval-Tree( x->right )
  switch ( n->op ) {
    case ‘+’ : return val1 + val2;
    case ‘-’ : return val1 - val2;
    case ‘*’ : return val1 * val2;
    case ‘/’ : return val1 / val2;
  }
```

General Trees: Design

• Children – in a linked list
Traversal of a general tree

Tree-Walk(x)
if x ≠ NULL then
  // pre-order operations
  foreach c in children(x)
    Tree-Walk(c)
  // post-order operations

Depth-first Traversal
• We note that each node could be visited twice in such a scheme
  — the first time the node is approached, and
  — the last time it is approached.

Pre-order Depth-first Traversal
• Visiting each node first results in the sequence
  A, B, C, D, E, F, G, H, I, J, K, L, M

Post-order Depth-first Traversal
• Visiting the nodes with their last visit:

Parenthesised tree serialisation
• Passing such a visitor results in the output:
  (A(B(C(D))(E(F(G))(H(I)(K)(L))(M))))
Breadth-First Traversal

- Breadth-first traversal would visit the nodes in the order:

Breadth-First Traversal

Breadth-First (x)
1 enqueue(Q, x)
2 while not empty(Q)
3 \( x = \text{dequeue}(Q) \)
4 print x->name // process node x
5 \( \text{foreach} \ c \text{ in next-child(x)} \)
6 \( \text{enqueue}(Q,c) \)

Printing Directories

- Given the directory structure

```
/  
  |  
usr/  
  |  
bin/  
  |  
local/  
  |  
var/  
  |  
  |  
adm/  
cron/  
  |  
log/  
```

Exercise

- Print the following statistics for a given (e.g. current working) directory:
  - subdirectory size (# of all subdirectories and files)
  - depth (maximal height)
  - width at all levels of depth...
  - maximal depth
  - largest directory in nr of subdirs and files in that directory
  - ...

Binary Search Tree (BST)

- MIT [link](http://ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-046JFall-2005/VideoLectures/detail/embed09.htm)

- Binary tree where values of the keys have a special order:
  \( \text{values(left subtree)} < \text{value(root)} <= \text{values(right subtree)} \)

Examples

- Here we see a complete binary search tree, and a binary search tree which is close to being complete -- balanced
Examples

- There are many different representations of the same ordered data:

   ![Diagram of tree structures]

Operations on dynamic sets

**SEARCH(S, k)**

A query that, given a set S and a key value k, returns a pointer x to an element in S such that key[x] = k, or NIL if no such element belongs to S.

**INSERT(S, x)**

A modifying operation that augments the set S with an element x. We usually assume that any fields in element x required by the set implementation have already been initialized.

**DELETE(S, x)**

A modifying operation that, given a pointer x to an element in the set S, removes x from S. (Note that this operation uses a pointer to an element x, not a key value.)

**MINIMUM(S)**

A query on a totally ordered set S that returns a pointer to the element of S with the smallest key.

**MAXIMUM(S)**

A query on a totally ordered set S that returns a pointer to the element of S with the largest key.

**SUCCESSOR(S, x)**

A query that, given an element x whose key is from a totally ordered set S, returns a pointer to the next larger element in S, or NIL if x is the maximum element.

**PREDECESSOR(S, x)**

A query that, given an element x whose key is from a totally ordered set S, returns a pointer to the next smaller element in S, or NIL if x is the minimum element.

Operations - search

**TREE-SEARCH (x, k)**

1. if x = NIL or k = x.key
2. then return x
3. if k < x.key
4. then return TREE-SEARCH(x.left, k)
5. else return TREE-SEARCH(x.right, k)

Iterative search

**ITERATIVE-TREE-SEARCH (x, k)**

1. while x ≠ NIL and k ≠ x.key
2. if k < x.key
3. then x = x.left
4. else x = x.right
5. return x

(Tail) Recursion "unrolling" – should be more efficient

Min and Max

**Tree-Minimum (x)**

1. while left[x] ≠ NIL
2. x = left[x]
3. return x

**Tree-Maximum (x)**

1. while right[x] ≠ NIL
2. x = right[x]
3. return x

Successor

**Tree-Successor (x)**

1. if right[x] ≠ NIL
2. then return Tree-Minimum(right[x])
3. y = parent[x]
4. while y ≠ NIL and x = right[y]
5. x = y; y = parent[y]
6. return y
Insert a node

- Find such a node where “next” position is missing...

Remove

- Suppose we wish to remove a node
- There are three situations: the node being removed
  - is a leaf node,
  - has exactly one child, or
  - is a full node (two children).

Remove

- If it is a leaf node, we can remove it:

Remove

- If the node has only one child, we can promote that child (with all the subtree underneath):

Remove

- If it is a full node, we copy the minimum element from the right sub-tree
- Recursively delete the value we copied

Example

- Consider the following tree
- We will twice remove the root
Example

• First, to remove 15, it is a full node
• We find the minimum element in the right sub-tree

Example

• We promote 42 to the root
• Proceed to remove 42 from the right sub-tree

Example

• This has one child, so we promote the entire sub-tree to replace 42

Example

• The root has been deleted, and the result is still a binary search tree

Example

• Next, let us remove 42
• Once again, it is a full node, so get the minimum element in the right sub-tree

Example

• We promote 45 to the root and proceed to delete 45 from the right sub-tree
Example

- The node 45 is a leaf node, so we may simply remove it.

Example

- Thus, the final tree, having removed 15 and then 42 is.

Reading

- CLRS: Binary Search Trees
- Visualisations:

Complexity...

- (Almost) all operations depend on the depth of the tree (or node affected)
- Binary search tree can get unbalanced, depth $O(n)$
- How to ensure this does not happen?

Balance

- If elements are added in random, tree is “automatically balanced” on average
- Otherwise: we must re-balance it ourselves...

[Diagram of balanced and unbalanced trees]
Balanced Binary Search Trees

- AVL trees
- 2-3 trees
- 2-3-4 trees
- B-trees
- Red-black trees

AVL-trees

- Adelson-Velskii and Landis
- In an AVL tree, the heights of the two child subtrees of any node **differ by at most one**;
- The AVL tree is named after its two inventors, G.M. Adelson-Velsky and E.M. Landis, who published it in their 1962 paper "An algorithm for the organization of information."

In an AVL tree, the **heights** of the two child subtrees of any node differ by at most one;
- Difference: -1, 0, 1
- Re-balance using rotations when getting out of balance...
- O(\(\log n\)) normal operations
- Up to O(\(\log n\)) re-balancing operations of O(1)
- an AVL tree's height is limited to 1.44 \(\log n\)

**Height of an AVL Tree**

- If \(n = 88\), the worst- and best-case scenarios differ in height by only 2:

**Height of an AVL Tree**

- If \(n = 10^6\), the bounds on \(h\) are:
  - The minimum height: \(\log_2(10^6) - 1 = 19\)
  - the maximum height: \(\log_2(10^6 / 1.8944) < 28\)

**Re-balancing**

- Can be done during each insertion-deletion
- AVL, Red-Black, ...
  - or each lookup (e.g. Splay trees)
  - Special re-balancing processes when computer otherwise idle
Red-Black trees

1. A node is either red or black.
2. The root is black. (This rule is used in some definitions and not others. Since the root can always be changed from red to black but not necessarily vice-versa this rule has little effect on analysis.)
3. All leaves are black.
4. Both children of every red node are black.
5. Every simple path from a node to a descendant leaf contains the same number of black nodes.

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Red-black trees

This data structure requires an extra one-bit color field in each node.

Red-black properties:
1. Every node is either red or black.
2. The root and leaves (NIL’s) are black.
3. If a node is red, then its parent is black.
4. All simple paths from any node \( x \) to a descendant leaf have the same number of black nodes = black-height(\( x \)).

Example of a red-black tree

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Height of a red-black tree

Theorem. A red-black tree with \( n \) keys has height \( h \leq 2 \lg(n + 1) \).

Proof. (The book uses induction. Read carefully.)

Intuition:
• Merge red nodes into their black parents.

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**Intuition:**
- Merge red nodes into their black parents.
- This process produces a tree in which each node has 2, 3, or 4 children.
- The 2-3-4 tree has uniform depth $h'$ of leaves.

Proof (continued)

- We have $h' \geq h/2$, since at most half the leaves on any path are red.
- The number of leaves in each tree is $n + 1$.
$\Rightarrow n + 1 \geq 2^{h'}$
$\Rightarrow \lg(n + 1) \geq h' \geq h/2$
$\Rightarrow h \leq 2 \lg(n + 1)$.

Query operations

**Corollary.** The queries SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR all run in $O(\lg n)$ time on a red-black tree with $n$ nodes.

Modifying operations

The operations INSERT and DELETE cause modifications to the red-black tree:
- the operation itself,
- color changes,
- restructuring the links of the tree via "rotations".

Rotations

Rotations maintain the inorder ordering of keys:
- $a \in \alpha, b \in \beta, c \in \gamma \Rightarrow a \leq b \leq \beta \leq c$.
- A rotation can be performed in $O(1)$ time.

Insertion into a red-black tree

**IDEA:** Insert $x$ in tree. Color $x$ red. Only red-black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

**Example:**
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### Pseudocode

```
RB-INSERT(T, x)
Tree-INSERT(T, x)
    color[x] ← RED  \> only RB property 3 can be violated
    while x ≠ root(T) and color[p[x]] = RED
        do if p[x] = left[p[x]]
            then y ← right[p[x]]  \> y = aunt/uncle of x
                if color[y] = RED
                    then (Case 1)
                        else if x = right[p[x]]
                            then (Case 2)  \> Case 2 falls into Case 3
                                (Case 3)
                            else (“then” clause with “left” and “right” swapped)
                                color[root(T)] ← BLACK
```

Graphical notation

Let ♦ denote a subtree with a black root.

All ♦'s have the same black-height.
• Suppose that $A$ and $D$, respectively were swapped.
• In both these cases, we perform similar rotations as before, and we are finished.

**Analysis**

- Go up the tree performing Case 1, which only recolors nodes.
- If Case 2 or Case 3 occurs, perform 1 or 2 rotations, and terminate.

**Running time:** $O(\log n)$ with $O(1)$ rotations.

**RB-DELETE** — same asymptotic running time and number of rotations as RB-INSERT (see textbook).
Bottom-Up Insertions

- If, at the end, the root is red, it can be coloured black

```
\begin{center}
\includegraphics[width=0.5\textwidth]{insertions.png}
\end{center}
```

Figure 1: The four cases for balancing a red-black tree.

Next we color the new node. If we color it red, we risk violating the first color invariant. If we color it black, we risk violating the second color invariant. However, note that the first invariant is a local property and the second invariant is a global property. In the hope that a local property will be easier to fix than a global property, we color the new node red.

If the parent of the new node is also red, then we have violated the first color invariant and need to rearrange and recolor the tree to restore the invariant. This is where rotations are used in ordinary red-black trees. Instead of rotations we use the following balancing transformation:

Take the red child, the red parent, and the (black) grandparent and locally balance these three nodes by making the smallest and largest nodes children of the middle node. Then color the middle node red and the other two nodes black. The middle node is linked back into the tree in place of the former black grandparent.

This balancing transformation is illustrated in Figure 1.

Complexity

- Rebalancing may need to rebalance the whole path up to the root => $O(\log n)$

```
\begin{center}
\includegraphics[width=0.5\textwidth]{complexity.png}
\end{center}
```

Other ideas

- Balancing can be an independent process — at night?

- Many search&insert&delete processes, and few rebalancing processes

- Local locking. Must ensure no deadlocks occur!
Properties of Red-Black trees

- No overhead for searching – efficient
- 100-200 lines of code, many symmetric cases
- Left-Leaning Red-Black trees (LLRB)

2-3, 2-3-4, B-trees

- Binary trees are useful for memory-based data structures
- Large databases and disk based systems would benefit of fewer reads of larger block sizes
- Organise data in a search tree that minimizes disk accesses

B-tree (m-way)

- $h = O(\log_m n)$
- In practice: 3-5 accesses to disk
**B-tree properties**
A B-tree of order $m$ (the maximum number of children for each node) is a tree which satisfies the following properties:

- Every node has at most $m$ children.
- Every node (except root and leaves) has at least $\lceil m/2 \rceil$ children.
- The root has at least two children if it is not a leaf node.
- All leaves appear in the same level, and carry information.
- A non-leaf node with $k$ children contains $k-1$ keys

**Half-full property ensures that ...**

- Two half-full nodes can be joined to make a legal node, and one full node can be split into two legal nodes (if there is room to push one element up into the parent).

**Example:** Two-level Insertion

- Inserting 29
- Leaf node is full, so we split it into two

**Example:** Two-level Insertion

- Parent node is full, so we must split it

**Example:** Two-level Insertion

- The root node must be updated

**Example:** Root Insertion

- Insert 67
- Leaf is full, so split it into two
Example: Root Insertion

• Parent is full, so split it into two

Example: Root Insertion

• Root is full, so split it into two

Example: Root Insertion

• Create a new root node

Example: Root Insertion

• The creators of the B-tree structure, Rudolf Bayer and Ed McCreight, have not explained what, if anything, the B stands for. Douglas Comer suggests a number of possibilities:
  • "Balanced," "Broad," or "Bushy" might apply [since all leaves are at the same level]. Others suggest that the "B" stands for Boeing [since the authors worked at Boeing Scientific Research Labs in 1972]. Because of his contributions, however, it seems appropriate to think of B-trees as "Bayer"-trees.[1]

Analogy between R-B and B-trees

• Multi-dimensional data
  – 2-dim (x,y)
  – 3D (x,y,z)
  – d-dim (x₁,...,xₙ)

• Does a point belong to a set?
• What is the closest point? (other data structures)
• ...

k-d tree

• Multi-dimensional data
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• Does a point belong to a set?
• What is the closest point? (other data structures)
• ...
**kd-Trees**

- Suppose we wish to partition the following points in a 2-dimensional kd-tree:

  - (0.01, 0.48), (0.02, 0.91), (0.02, 0.97), (0.03, 0.90),
  - (0.04, 0.41), (0.04, 0.61), (0.04, 0.61), (0.05, 0.07),
  - (0.05, 0.88), (0.05, 0.97), (0.06, 0.05), (0.06, 0.28),
  - (0.08, 0.89), (0.09, 0.55), (0.23, 0.23),
  - (0.33, 0.07), (0.37, 0.04), (0.41, 0.89), (0.46, 0.58),
  - (0.54, 0.65), (0.55, 0.02), (0.55, 0.54), (0.56, 0.78),
  - (0.68, 0.42), (0.73, 0.69), (0.74, 0.97), (0.94, 0.01),
  - (0.95, 0.07), (0.97, 0.18)

- The first step is to order the points based on the 1st coordinate and find the median:

- The median point, (0.29, 0.15), forms the root of our kd-tree

- This partitions the remaining points into two sets:
**kd-Trees**

- Starting with the first partition, we order these according to the 2\(^{nd}\) coordinate:

  - (0.04, 0.03), (0.04, 0.06), (0.05, 0.07), (0.23, 0.23),
  - (0.23, 0.23), (0.24, 0.41), (0.31, 0.48), (0.33, 0.48),
  - (0.44, 0.44), (0.50, 0.89), (0.62, 0.94), (0.82, 0.87),
  - (0.85, 0.88), (0.89, 0.90), (0.82, 0.97).

- This point creates the left child of the root.

- StarZng with the second partition, we also order these according to the 2\(^{nd}\) coordinate:

  - (0.11, 0.23), (0.16, 0.23), (0.27, 0.42), (0.33, 0.42),
  - (0.43, 0.43), (0.57, 0.67), (0.71, 0.67), (0.84, 0.64),
  - (0.51, 0.54), (0.60, 0.58), (0.54, 0.64), (0.73, 0.49),
  - (0.54, 0.78), (0.64, 0.80), (0.74, 0.97).

- This point creates the right child of the root.

- Next, ordering the partitioned elements by the 1\(^{st}\) coordinate, we choose the medians to find the children of the left child (0.09, 0.55):

  - (0.01, 0.48), (0.04, 0.46), (0.04, 0.44), (0.45, 0.07),
  - (0.46, 0.05), (0.50, 0.20), (0.23, 0.23),
  - (0.02, 0.01), (0.02, 0.97), (0.03, 0.96), (0.44, 0.42),
  - (0.04, 0.02), (0.05, 0.97), (0.08, 0.98).

- Doing the same with the two right partitions, we get the children of the right child of the root:

  - (0.33, 0.07), (0.37, 0.06), (0.35, 0.02), (0.34, 0.02),
  - (0.50, 0.07), (0.57, 0.06), (0.87, 0.18),
  - (0.41, 0.80), (0.46, 0.80), (0.54, 0.49), (0.55, 0.54),
  - (0.56, 0.78), (0.73, 0.66), (0.74, 0.97).
kd-Trees

At the next level, we order the points again based on the 2nd coordinate and choose the medians:

- (0.04, 0.02), (0.08, 0.04), (0.01, 0.06)
- (0.04, 0.05), (0.03, 0.39), (0.04, 0.26)
- (0.03, 0.03), (0.02, 0.88), (0.02, 0.01)
- (0.05, 0.88), (0.06, 0.88), (0.05, 0.07)
- (0.53, 0.02), (0.37, 0.46), (0.33, 0.07)
- (0.53, 0.02), (0.37, 0.46), (0.33, 0.07)
- (0.46, 0.04), (0.54, 0.46), (0.41, 0.08)
- (0.78, 0.04), (0.54, 0.78), (0.74, 0.07)

The result is a 2-dimensional kd-tree of the given 31 points.

Finally, the last point, a leaf node, falls within the given box.

A useful application of a kd-tree provides an efficient data structure for counting the number of points which fall within a given $k$-dimensional rectangle.

This is used in image processing: locating objects within a scene, ray tracing, etc.

Find the points which lie in the quadrant $[0.5, 1] \times [0, 0.5]$

The traversal rules we will follow are:
- we always match the coordinate corresponding to the level we are current at
- if that coordinate is less than the corresponding interval of the box, we only need to visit the right sub-tree
- if that coordinate is greater than the corresponding interval, we need only visit the left sub-tree
- otherwise, we check if the root is in the box and we visit both sub-trees
$kd$-Trees

- Starting with the left sub-tree: $0.94 \in [0.5, 1]$
- We note that $(0.94, 0.02) \in [0.5, 1] \times [0, 0.5]$ and we visit both sub-trees

$k$-trees are not suitable for efficiently finding the nearest neighbour in high dimensional spaces.

- As a general rule, if the dimensionality is $D$, then number of points in the data, $N$, should be $N \gg 2^D$.
- Otherwise, when $kd$-trees are used with high-dimensional data, most of the points in the tree will be evaluated and the efficiency is no better than exhaustive search.
- The problem of finding NN in high-dimensional data is thought to be NP-hard \cite{2}, and approximate nearest-neighbour methods are used instead.

**See also** (Wikipedia)

- implicit $kd$-tree
- min/max $kd$-tree
- Quadtree
- Octree
- Bounding Interval Hierarchy
- Nearest neighbor search
- Klee’s measure problem
- $kd$-trie

**Quadtree**

- 2-dimensional
- 4 quadrants
- Either point or area based

**Octree – 3D, 8 children**
R-Tree

Variants

• R, R+, R*

Difference between R+ trees and R trees
R+ trees are a compromise between R-trees and kd-trees; they avoid overlapping of internal nodes by inserting an object into multiple leaves if necessary.
R+ trees differ from R trees in that:
• Nodes are not guaranteed to be at least half filled
• The entries of any internal node do not overlap
• An object ID may be stored in more than one leaf node

Advantages
Because nodes are not overlapped with each other, point query performance benefits since all spatial regions are covered by at most one node. A single path is followed and fewer nodes are visited than with the R-tree

Priority queue

• Insert Q, x

• Retrieve next x from Q s.t. x.value is largest

• Sorted list implementation:
  – O(n) to insert x into right place
  – O(1) access, O(1) delete

Binary heap

Complete – missing nodes only at the lowest level

Heap property – on any path parent has higher priority
Typically: min-heaps

Priority queue
insert ( Q, x )
pop Q

Binary heap - Insert

Insert into a next allowed place
Make sure heap property is restored
Binary heap – Insert – “Bubble up”

Use Array based implementation

Insert

```
insert( int A[], int x, int *last) {
    (*last)++ ;
    A[*last] = x;
    bubbleUp( A, *last ) ;
}
```

Bubble up

```
BubbleUp( int A[], int i)
while ( ( i>1 ) && A[i] > A[i/2] ){
    swap( A, i, i/2 ) ;
    i=i/2;
}
```

Delete (max)

- Remove top value (make free space)
- Remove last element
- Insert to top value & bubble down to rightful place
Binary heap – Delete – “Bubble down”

Cost
- Insert – $O(\log n)$
- Delete – $O(\log n)$

Heap-sort
- Heapsify the array
- while not empty
  - pop_largest
  - copy to next free place

1. remove last (x), decrease size,
2. copy largest to new free place
3. insert x to L[1], bubble down

Build heap
- n times insert to heap = $O(n \log n)$

“Sort”
- n times repeat remove largest = $O(n \log n)$

Total: $O(n \log n)$ method

Heapify... in linear time

- last n/2 – ignore
- n/4 - bubble down (at most by 1 level)
- n/8 – bubble down (at most by 2 levels)
- ...

$$\sum_{i=1}^{\log n} \frac{i}{2^i} \leq \frac{n}{2} \sum_{i=1}^{n} \frac{i}{2^i}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$$
$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{1}{2}$$
$$\frac{1}{8} + \frac{1}{16} + \ldots = \frac{1}{4}$$
... Sum = 2
**Introduction to Algorithms**

**LECTURE 11**

Augmenting Data Structures
- Dynamic order statistics
- Methodology
- Interval trees

Prof. Charles E. Leiserson

---

**Dynamic order statistics**

OS-SELECT\(i, S\): returns the \(i\)th smallest element in the dynamic set \(S\).

OS-RANK\(x, S\): returns the rank of \(x \in S\) in the sorted order of \(S\)'s elements.

**IDEA:** Use a red-black tree for the set \(S\), but keep subtree sizes in the nodes.

**Notation for nodes:**

\[
\begin{array}{c|c}
\text{key} & \text{size} \\
\hline
\end{array}
\]

---

**Example of an OS-tree**

\[
\begin{array}{c|c|c}
\text{C} & \text{5} & \text{M} \\
\text{A} & \text{1} & \text{D} \\
\text{D} & \text{1} & \text{H} \\
\text{H} & \text{1} & \\
\end{array}
\]

\[\text{size}[x] = \text{size}[\text{left}[x]] + \text{size}[\text{right}[x]] + 1\]

---

**Selection**

**Implementation trick:** Use a *sentinel* (dummy record) for NIL such that \(\text{size}[\text{NIL}] = 0\).

OS-SELECT(x, i) \(\triangleright\) \(i\)th smallest element in the subtree rooted at \(x\)

\[
k \leftarrow \text{size}[\text{left}[x]] + 1 \quad \triangleright \quad k = \text{rank}(x)
\]

if \(i = k\) then return \(x\)

if \(i < k\)

then return OS-SELECT(left[x], i)

else return OS-SELECT(right[x], i - k)

(\(\text{OS-RANK}\) is in the textbook.)

---

**Example**

OS-SELECT(root, 5)

\[
\begin{array}{c|c|c|c|c}
\text{M} & \text{9} & \text{i - 5} & \text{k - 6} \\
\text{A} & \text{1} & \text{i - 5} & \text{k - 2} \\
\text{D} & \text{1} & \text{i - 1} & \text{k - 1} \\
\end{array}
\]

Running time = \(O(h) = O(\log n)\) for red-black trees.

---

**Data structure maintenance**

**Q.** Why not keep the ranks themselves in the nodes instead of subtree sizes?

**A.** They are hard to maintain when the red-black tree is modified.

**Modifying operations:** 

**INSERT** and **DELETE**.

**Strategy:** Update subtree sizes when inserting or deleting.

---

29.2.2012
In computer science, an interval tree, also called a segment tree or segtree, is an ordered tree data structure to hold intervals. Specifically, it allows one to efficiently find all intervals that overlap with any given interval or point. It is often used for windowing queries, for example, to find all roads on a computerized map inside a rectangular viewport, or to find all visible elements inside a three-dimensional scene.

The trivial solution is to visit each interval and test whether it intersects the given point or interval, which requires \( \Theta(n) \) time, where \( n \) is the number of intervals in the collection. Since a query may return all intervals, for example if the query is a large interval intersecting all intervals in the collection, this is asymptotically optimal; however, we can do better by considering output-sensitive algorithms, where the runtime is expressed in terms of \( m \), the number of intervals produced by the query.
New operations

4. Develop new dynamic-set operations that use the information.

INTERVAL-SEARCH(i)

\[ x \leftarrow \text{root} \]

while \( x \neq \text{NIL} \) and \( \text{low}[i] > \text{high}[\text{int}[x]] \)

or \( \text{low}[\text{int}[x]] > \text{high}[i] \)

do \( i \) and \( \text{int}[x] \) don't overlap

if \( \text{left}[x] \neq \text{NIL} \) and \( \text{low}[i] \leq \text{m}[\text{left}[x]] \)

then \( x \leftarrow \text{left}[x] \)

else \( x \leftarrow \text{right}[x] \)

return \( x \)
**Example 2: INTERVAL-SEARCH([12,14])**

- $x \leftarrow \text{root}$
- $[12,14]$ and $[17,19]$ don’t overlap
- $12 \leq 18 \Rightarrow x \leftarrow \text{left}(x)$

- $x = \text{left}(x)$
- $[12,14]$ and $[5,11]$ don’t overlap
- $12 > 8 \Rightarrow x \leftarrow \text{right}(x)$

- $x = \text{right}(x)$
- $[12,14]$ and $[15,18]$ don’t overlap
- $12 > 10 \Rightarrow x \leftarrow \text{right}(x)$

**Analysis**

- Time = $O(h) = O(\lg n)$, since INTERVAL-SEARCH does constant work at each level as it follows a simple path down the tree.
- List all overlapping intervals:
  - Search, list, delete, repeat.
  - Insert them all again at the end.
- Time = $O(k \lg n)$, where $k$ is the total number of overlapping intervals.
- This is an output-sensitive bound.
- Best algorithm to date: $O(k + \lg n)$.

**Correctness**

**Theorem.** Let $L$ be the set of intervals in the left subtree of node $x$, and let $R$ be the set of intervals in $x$’s right subtree.

- If the search goes right, then
  - $\{ i' \in L : i'$ overlaps $i$ $\} = \emptyset$.
- If the search goes left, then
  - $\{ i' \in L : i'$ overlaps $i$ $\} = \emptyset$.
  - $\{ i' \in R : i'$ overlaps $i$ $\} = \emptyset$.

In other words, it's always safe to take only 1 of the 2 children: we'll either find something, or nothing was to be found.
Correctness proof

Proof. Suppose first that the search goes right.
- If $\text{left}[x] = \text{NIL}$, then we’re done, since $L = \emptyset$.
- Otherwise, the code dictates that we must have $\text{low}[i] > \text{m}[\text{left}[x]]$. The value $\text{m}[\text{left}[x]]$ corresponds to the high endpoint of some interval $j \in L$, and no other interval in $L$ can have a larger high endpoint than $\text{high}[j]$.

\[ \text{high}[j] = \text{m}[\text{left}[x]] \]

- Therefore, $\{i' \in L : i' \text{ overlaps } i\} = \emptyset$.

Left:

Proof (continued)

Suppose that the search goes left, and assume that $\{i' \in L : i' \text{ overlaps } i\} = \emptyset$.
- Then, the code dictates that $\text{low}[i] \leq \text{m}[\text{left}[x]] = \text{high}[j]$ for some $j \in L$.
- Since $j \in L$, it does not overlap $i$, and hence $\text{high}[i] < \text{low}[j]$.
- But, the binary-search-tree property implies that for all $i' \in R$, we have $\text{low}[j] \leq \text{low}[i']$.
- But then $\{i' \in R : i' \text{ overlaps } i\} = \emptyset$. 

Left:

Combining info: Treap

- Heap and binary search tree properties together
- Treap
Combining info

- insert 18, priority 20

- delete max priority...

- delete max priority...

- delete max priority...

Treap

- Can be used to make a random-like tree: priorities can be assigned by random, unique values...


- In computer science, a treap is a binary search tree that orders the nodes by adding a random priority attribute to a node, as well as a key. The nodes are ordered so that the keys form a binary search tree and the priorities obey the max heap order property. The name treap is a portmanteau of tree and heap.

- A portmanteau word (pronounced /pɔːtˈmæntəʊ/ (help·info)) is used broadly to mean a blend of two (or more) words, and narrowly in linguistics fields to mean only a blend of two or more function words.
### Union-find

- **Domain** \( X = \{ x_1, \ldots, x_n \} \)
- \( x_i \) belongs to a set \( S_j \)
- Non-intersecting sets.
- Union of sets: \( S_i = S_i \cup S_j \)
- Which set \( S_i \) does an element \( x_j \) belong to?

#### Link elements until “0”

<table>
<thead>
<tr>
<th>Set</th>
<th>0</th>
<th>0</th>
<th>3</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>6</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

**Union 4, 6**

<table>
<thead>
<tr>
<th>Set</th>
<th>0</th>
<th>0</th>
<th>4</th>
<th>5</th>
<th>4</th>
<th>7</th>
<th>4</th>
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<td>8</td>
</tr>
</tbody>
</table>

**Find** 6

<table>
<thead>
<tr>
<th>Set</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>3</th>
<th>0</th>
<th>3</th>
<th>0</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
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<td>6</td>
<td>7</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

**Find** 6

<table>
<thead>
<tr>
<th>Set</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>3</th>
<th>0</th>
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<td>7</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

**Sets** = \( \{1\}, \{2\}, \ldots, \{n\} \)

- Non-overlapping, each value belongs to a set
- Merge sets \( i, j \) (give new set id \( i \), remove \( j \))
  - Union
- Which set does \( x \) belong to?
  - Find

- At every find – “flatten the tree”
Union-Find
A data structure for maintaining a collection of disjoint sets

Course: Data Structures
Lecturer: Uri Zwick
March 2008

Union-Find

- Make(x): Create a set containing x
- Union(x, y): Unite the sets containing x and y
- Find(x): Return a representative of the set containing x

Union Find

<table>
<thead>
<tr>
<th>make</th>
<th>union</th>
<th>find</th>
</tr>
</thead>
<tbody>
<tr>
<td>O(1)</td>
<td>O(α(n))</td>
<td>O(α(n))</td>
</tr>
</tbody>
</table>

Fun applications: Generating mazes

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>make(1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>make(2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>make(16)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>find(6)=find(7) ?</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>union(6,7)</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>find(7)=find(11) ?</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>union(7,11)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Choose edges in random order and remove them if they connect two different regions

Fun applications: Generating mazes

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<th>3</th>
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</tr>
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<td>7</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

Generating mazes – a larger example

Construction time -- $O(n^2 \alpha(n^2))$
http://biit.cs.ut.ee/~vilo/Algorithmics/maze.cgi

More serious applications:

- Maintaining an equivalence relation
- Incremental connectivity in graphs
- Computing minimum spanning trees
- ...

Union Find

Represent each set as a rooted tree

Union by rank

Path compression

The parent of a vertex $x$ is denoted by $p[x]$
Find($x$) traces the path from $x$ to the root

Union by rank

A tree of rank $r$ contains at least $2^r$ elements
If $x$ is not a root, then $\text{rank}(x) = \text{rank}(p[x])$

Path Compression

Union Find - pseudocode

```plaintext
Function make-set(x)
    p[x] ← x
    rank[x] ← 0

Function union(x, y)
    link(find(x), find(y))

Function link(x, y)
    if rank[x] > rank[y] then
        p[y] ← x
    else
        p[x] ← y
        if rank[x] = rank[y] then
            rank[y] ← rank[y] + 1

Function find(x)
    if p[x] ≠ x then
        p[x] ← find(p[x])
    return p[x]
```
Union-Find

<table>
<thead>
<tr>
<th>make</th>
<th>link</th>
<th>find</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(\log n)$</td>
</tr>
</tbody>
</table>

Amortized

<table>
<thead>
<tr>
<th>make</th>
<th>link</th>
<th>find</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(1)$</td>
<td>$O(\alpha(n))$</td>
<td>$O(\alpha(n))$</td>
</tr>
</tbody>
</table>

Nesting / Repeated application

$$f^{(i)}(n) = f(f(\ldots(f(n))\ldots))$$

$i$ times

$$f^{(0)}(n) = n$$

$$f^{(i)}(n) = f(f^{(i-1)}(n))$$, for $i > 0$

- $f(n) = n + 1$
- $f(n) = 2n$
- $f(n) = 2^n$
- $f(n) = \log n$

- $f^{(5)}(n) = n + 5$
- $f^{(7)}(n) = 2^7 n$
- $f^{(3)}(n) = 2^{2^3 n}$
- $f^{(2)}(n) = \log\log n$

Ackermann’s function

$$A_k(n) = \begin{cases} 
  n + 1 & \text{if } k = 1, \\
  A_{k-1}^{(n+1)}(n) & \text{if } k > 1.
\end{cases}$$

- $A_1(n) = n + 1$
- $A_2(n) = 2n + 1$
- $A_3(n) = 2^{2^{n+1} + (n+1) - 1}$
- $A_4(n)$ is ?

Ackermann’s function (modified)

$$A_k(n) = \begin{cases} 
  n + 1 & \text{if } k = 1, \\
  A_{k-1}^{(n+1)}(n) & \text{if } k > 1.
\end{cases}$$

- $\bar{A}_1(n) = 2n$
- $\bar{A}_2(n) = 2^n$
- $\bar{A}_3(n) = \underbrace{2^{2^{\ldots^{2}}}}_{n}$
- $\bar{A}_4(n) = tower(n) = 2^{2^{\ldots^{2}}}$

Inverse functions

- $F(n) \implies f(n) = \min\{k \geq 1 \mid F(k) \geq n\}$
- $F(n) = n + 1$  $f(n) = n - 1$
- $F(n) = 2n$  $f(n) = \lfloor \frac{n}{2} \rfloor$
- $F(n) = 2^n$  $f(n) = \lceil \log_2 n \rceil$
- $F(n) = tower(n)$  $f(n) = \log^* n$

Inverse Ackermann function

$$\alpha_r(n) = \min\{k \geq 1 \mid A_k(r) \geq n\}$$

- $\alpha(n) = \alpha_1(n) = \min\{k \geq 1 \mid A_k(1) \geq n\}$
- $\alpha(n)$ is the inverse of the function $A_n(1)$

- $A_n(1) = A_{n-1}^{(2)}(1) = A_{n-1}(A_{n-1}(1)) > A_{n-1}(n)$

The first "column"

A "diagonal"
Level and Index

Back to union-find...

\[ p[x] \]

If \( p[x] \neq x \) and \( \text{rank}[x] > 0 \):

\[
\begin{align*}
\text{level}(x) &= \max \{ k \geq 1 \mid A_k(\text{rank}[x]) \leq \text{rank}[p[x]] \} \\
\text{index}(x) &= \max \{ i \geq 1 \mid A_{\text{level}(x)}^{(i)}(\text{rank}[x]) \leq \text{rank}[p[x]] \}
\end{align*}
\]

If \( \text{p}[x] = x \) or \( \text{rank}[x] = 0 \):

\[
\begin{align*}
\text{level}(x) &= \text{index}(x) = 0
\end{align*}
\]

Potentials

\[
\phi(x) = (\alpha(n) - \text{level}(x)) \cdot \text{rank}[x] - \text{index}(x)
\]

\[
\Phi = \sum_x \phi(x)
\]

If \( p[x] \neq x \) and \( \text{rank}[x] > 0 \):

\[
\begin{align*}
1 &\leq \text{level}(x) < \alpha(n) \\
1 &\leq \text{index}(x) \leq \text{rank}[x]
\end{align*}
\]

\[
\phi(x) \geq 0
\]

Definition on level

If \( p[x] \neq x \) and \( \text{rank}[x] > 0 \):

\[
\text{level}(x) = \max \{ k \geq 1 \mid A_k(\text{rank}[x]) \leq \text{rank}[p[x]] \}
\]

If \( p[x] \neq x \) and \( \text{rank}[x] > 0 \):

\[
1 \leq \text{level}(x) < \alpha(n)
\]

\[
\begin{align*}
A_1(\text{rank}[x]) &= \text{rank}[x] + 1 \leq \text{rank}[p[x]] \\
A_{\alpha(n)}(\text{rank}[x]) &\geq A_{\alpha(n)}(1) \geq n > \text{rank}[p[x]]
\end{align*}
\]

Bounds on index

If \( p[x] \neq x \) and \( \text{rank}[x] > 0 \):

\[
\text{index}(x) = \max \{ i \geq 1 \mid A_{\text{level}(x)}^{(i)}(\text{rank}[x]) \leq \text{rank}[p[x]] \}
\]

If \( p[x] \neq x \) and \( \text{rank}[x] > 0 \):

\[
1 \leq \text{index}(x) \leq \text{rank}[x]
\]

\[
\begin{align*}
A_{\text{level}(x)}^{(1)}(\text{rank}[x]) &= A_{\text{level}(x)}(\text{rank}[x]) \leq \text{rank}[p[x]] \\
A_{\text{level}(x)}^{(\text{rank}[x]+1)}(\text{rank}[x]) &= A_{\text{level}(x)+1}(\text{rank}[x]) > \text{rank}[p[x]]
\end{align*}
\]

Amortized cost of make

Actual cost: \( O(1) \)
\[ \Delta \Phi: \quad 0 \]
Amortized cost: \( O(1) \)

Amortized cost of link

Actual cost: \( O(1) \)

The potentials of \( y \) and \( z_1, \ldots, z_k \) can only decrease

The potentials of \( x \) is increased by at most \( \alpha(n) \)

\[ \Delta \Phi \leq \alpha(n) \]

Actual cost: \( O(\alpha(n)) \)
Amortized cost of find

Suppose that:

\[ 0 < i < j < \ell \]
\[ \text{level}(x_i) = \text{level}(x_j) \]

\[ A^{(\text{index}(x_i)+1)}(\text{rank}[x_i]) \]
\[ = A^{\text{level}(x_i)}(A^{(\text{index}(x_i)+1)}(\text{rank}[x_i])) \]
\[ \leq A^{\text{level}(x_j)}(\text{rank}[p[x_i]]) \]
\[ \leq A^{\text{level}(x_j)}(\text{rank}[x_j]) \]
\[ \leq \text{rank}[p[x_i]] \]
\[ \leq \text{rank}[x_j] \]

\[ \Delta \Phi \text{ is decreased} \]

Amortized cost of find

If \( \text{level}(x) \) is unchanged, then \( \text{index}(x) \) is either unchanged or is increased.

If \( \text{level}(x) \) is increased, then \( \text{index}(x) \) is decreased by at most \( \text{rank}[x] - 1 \).

**Amortized cost:** \( \alpha(n) + 1 \)

**Actual cost:**

\[ l + 1 \]
\[ \Delta \Phi \leq (\alpha(n) + 1) - (l + 1) \]

Amortized cost:

\( \alpha(n) + 1 \)