Advanced Algorithmics (6EAP)  
**Graphs I**  
Jaak Vilo  
2012 Spring

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CLRS: Chapter 22  
Elementary Graph Algorithms

Some slides from:  
http://www.cc.nctu.edu.tw/~claven/course/Algorithm/

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Lecture 10:  
Graph Data Structures  
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Introduction

- \( G = (V, E) \)
  - \( V \) = vertex set (nodes)
  - \( E \) = edge set (arcs)
- Graph representation
  - Adjacency list
  - Adjacency matrix
- Graph search
  - Breadth-first search (BFS)
  - Depth-first search (DFS)
    - Topological sort
    - Strongly connected components

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Road Networks

In modeling a road network, the vertices may represent the cities or junctions, certain pairs of which are connected by roads/edges.
Electronic Circuits

In an electronic circuit, with junctions as vertices and components as edges.

A Simple Metabolic Pathway

Metabolic Regulation - Methionine Biosynthesis in E. coli

Soviet Rail Network, 1955

Find a shortest path from station A to station B.

-need serious thinking to get a correct algorithm.

Evolutionary relationships among organisms based on similarity of the primary sequences of their CYTOCHROME c proteins

Common ancestor

Green arrows - upregulation
Red arrows - downregulation
Thickness of arrow represents certainty of direction (up/down)
A complete graph

Filter
- choose a list of genes
  (MATING, marked in red)
- filter for these genes plus
  neighbouring genes from the graph

Mutation network $\Delta_{y2}$

Mutation network $\Delta_{y4}$

Phyladibo network 1.0.1.3 version 0.5.0.15, underpinned is green set of genes which are mouse homologues. The grey less connected assemblies then are dog and fish. Green nodes - these above attributes closely named genes function (light blue), mitochondrial function (mid blue), cleaver (blue, purple) and a group of genes of unknown function (unknown, grey)

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Graphs

- Set of nodes $|V| = n$
- Set of edges $|E| = m$
  - Undirected edge/graph: pairs of nodes $(v, w)$
  - Directed edge/graph: pairs of nodes $(v, w)$
- Set of neighbors of $v$: set of nodes connected by an edge with $v$ (directed: in-neighbors, out-neighbors)
- Degree of a node: number of its neighbors (indegree, outdegree)
- Path: a sequence of nodes such that every two consecutive nodes constitute an edge
- Length of a path: number of nodes minus 1
- Distance between two nodes: the length of the shortest path between these nodes
- Diameter: the longest distance in the graph

Choose

- The boss wants to produce programs to solve the following two problems
  - Euler circuit problem:
    - given a graph $G$, find a way to go through each edge exactly once.
  - Hamilton circuit problem:
    - given a graph $G$, find a way to go through each vertex exactly once.
- The two problems seem to be very similar.
- Person A takes the first problem and person B takes the second.
- Outcome: Person A quickly completes the program, whereas person B works 24 hours per day and is fired after a few months.
Euler Circuit: The original Konigsberg bridge

- Every vertex of this graph has an even degree, therefore there exists an Eulerian graph. Following the edges in alphabetical order gives an Eulerian circuit/cycle.

Hamilton Circuit

- Traveling salesman problem (TSP).

A joke (continued):

- Why? No body in the company has taken Algorithmics class.
- Explanation:
  - Euler circuit problem can be easily solved in polynomial time.
  - Hamilton circuit problem is proved to be NP-hard.
  - So far, no body in the world can give a polynomial time algorithm for a NP-hard problem.
  - Conjecture: there does not exist polynomial time algorithm for this problem.

“I can’t find an efficient algorithm, I guess I’m just too dumb.”

“I can’t find an efficient algorithm, because no such algorithm is possible!”
Flavors of Graphs

The first step in any graph problem is determining which flavor of graph you are dealing with. Learning to talk the talk is an important part of walking the walk. The flavor of graph has a big impact on which algorithms are appropriate and efficient.

Directed vs. Undirected Graphs

A graph \( G = (V, E) \) is undirected if edge \((x, y) \in E\) implies that \((y, x) \) is also in \( E \).

Road networks between cities are typically undirected. Street networks within cities are almost always directed because of one-way streets. Most graphs of graph-theoretic interest are undirected.

Weighted vs. Unweighted Graphs

In weighted graphs, each edge (or vertex) of \( G \) is assigned a numerical value, or weight.

The edges of a road network graph might be weighted with their length, drive-time or speed limit. In unweighted graphs, there is no cost distinction between various edges and vertices.

Simple vs. Non-simple Graphs

Certain types of edges complicate the task of working with graphs. A self-loop is an edge \((x, x)\) involving only one vertex. An edge \((x, y)\) is a multi-edge if it occurs more than once in the graph.

Any graph which avoids these structures is called simple.

Sparse vs. Dense Graphs

Graphs are sparse when only a small fraction of the possible number of vertex pairs actually have edges defined between them.

Graphs are usually sparse due to application-specific constraints. Road networks must be sparse because of road junctions. Typically dense graphs have a quadratic number of edges while sparse graphs are linear in size.
Cyclic vs. Acyclic Graphs

An acyclic graph does not contain any cycles. Trees are connected acyclic undirected graphs.

Directed acyclic graphs are called DAGs. They arise naturally in scheduling problems, where a directed edge \((x, y)\) indicates that \(x\) must occur before \(y\).

Implicit vs. Explicit Graphs

Many graphs are not explicitly constructed and then traversed, but built as we use them.

A good example arises in backtrack search.

Embedded vs. Topological Graphs

A graph is embedded if the vertices and edges have been assigned geometric positions.

Example: TSP or Shortest path on points in the plane.
Example: Grid graphs.
Example: Planar graphs.

Labeled vs. Unlabeled Graphs

In labeled graphs, each vertex is assigned a unique name or identifier to distinguish it from all other vertices.

An important graph problem is isomorphism testing, determining whether the topological structure of two graphs are in fact identical if we ignore any labels.

The Friendship Graph

Consider a graph where the vertices are people, and there is an edge between two people if and only if they are friends.

This graph is well-defined on any set of people: SUNY SB, New York, or the world.
What questions might we ask about the friendship graph?

If I am your friend, does that mean you are my friend?

A graph is undirected if \((x, y)\) implies \((y, x)\). Otherwise the graph is directed.
The “heard-of” graph is directed since countless famous people have never heard of me!
The “had-sex-with” graph is presumably undirected, since it requires a partner.
Perception of/and experience

- Simple example of 5 entities (persons) and their relationships
- Who would you prefer to be?
- Who wouldn’t you want to be?
- And what if the relationship means "company A sells to company B"?
- What if relationship means "love"?

Visualization and human computation "brain exercise"

Am I my own friend?
An edge of the form \((x, x)\) is said to be a loop. If \(x\) is \(y\)'s friend several times over, that could be modeled using multiple edges, multiple edges between the same pair of vertices. A graph is said to be simple if it contains no loops and multiple edges.

Am I linked by some chain of friends to the President?
A path is a sequence of edges connecting two vertices. Since Mel Brooks is my father’s-sister’s-husband’s cousin, there is a path between me and him!

How close is my link to the President?
If I were trying to impress you with how tight I am with Mel Brooks, I would be much better off saying that Uncle Lenny knows him than to go into the details of how connected I am to Uncle Lenny. Thus we are often interested in the shortest path between two nodes.

Is there a path of friends between any two people?
A graph is connected if there is a path between any two vertices. A directed graph is strongly connected if there is a directed path between any two vertices.

Who has the most friends?
The degree of a vertex is the number of edges adjacent to it.
Given a graph $G$, its line graph $L(G)$ is a graph such that
• each vertex of $L(G)$ represents an edge of $G$; and
• two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint ("are adjacent") in $G$.

**Complete graph**
- Every node is connected to every other node
- Clique – fully connected subgraph of a graph

**Subgraph**
- Subset of vertices $(m)$ $V'$ is a subset of $V$
- Subset of edges $(n)$ $E'$ is a subset of $E$, s.t. $(u,v)$ in $E'$ if $u,v$ both in $V$)
- Nr of different possible graphs of size $m,n$ is huge

**How many different subgraphs does a complete graph have?**
- How many?
  - $2^n$ different subsets of vertices (= many!)  
  - Likewise, nr of edges is any subset of set of edges...
- $5$ nodes => $5*4/2$ different possible undirected edges without self-loops
  - Calculate the possibility of each edge being present or not...
  - Directed: ->, <-, <->, none (4 options)

**Representation For An Undirected Graph**

Figure 23.1: Two representations of an undirected graph. (a) An undirected graph $G$ having five vertices and seven edges. (b) An adjacency list representation of $G$. (c) The adjacency matrix representation of $G$.
Representation of Graphs

- Adjacency list: $O(V+E)$
  - Preferred for sparse graph
  - $|E| << |V|^2$
  - Adj[u] contains all the vertices v such that there is an edge (u, v) $\in E$
  - Weighted graph: $w(u,v)$ is stored with vertex v in Adj[u]
  - No quick way to determine if a given edge is present in the graph

- Adjacency matrix: $O(V^2)$
  - Preferred for dense graph
  - Symmetry for undirected graph
  - Weighted graph: store $w(u,v)$ in the (u, v) entry
  - Easy to determine if a given edge is present in the graph

Tradeoffs Between Adjacency Lists and Adjacency Matrices

<table>
<thead>
<tr>
<th>Comparison</th>
<th>Adjacency List</th>
<th>Adjacency Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space</td>
<td>$O(V+E)$</td>
<td>$O(V^2)$</td>
</tr>
<tr>
<td>Access to edge (u, v)</td>
<td>O(1)</td>
<td>O(1)</td>
</tr>
<tr>
<td>Access to current degree</td>
<td>O(V)</td>
<td>O(V^2)</td>
</tr>
<tr>
<td>Less memory for small graphs</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Less memory for large graphs</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Easy insertion and deletion</td>
<td>O(V)</td>
<td>O(V^2)</td>
</tr>
<tr>
<td>Delete vertex from the list</td>
<td>O(V)</td>
<td>O(V^2)</td>
</tr>
<tr>
<td>Insert new vertex in the list</td>
<td>O(V)</td>
<td>O(V^2)</td>
</tr>
<tr>
<td>Insert new edge</td>
<td>O(V)</td>
<td>O(V^2)</td>
</tr>
</tbody>
</table>

Both representations are very useful and have different properties, although adjacency lists are probably better for most problems.

Traversing a Graph

One of the most fundamental graph problems is to traverse every edge and vertex in a graph. For efficiency, we must make sure we visit each edge at most twice. For correctness, we must do the traversal in a systematic way so that we don’t miss anything. Since a maze is just a graph, such an algorithm must be powerful enough to enable us to get out of an arbitrary maze.

Marking Vertices

The key idea is that we must mark each vertex when we first visit it, and keep track of what have not yet completely explored.
Each vertex will always be in one of the following three states:
- **undiscovered** – the vertex in its initial, virgin state.
- **discovered** – the vertex after we have encountered it, but before we have checked out all its incident edges.
- **processed** – the vertex after we have visited all its incident edges.

Breadth-First Search (BFS)

- Graph search: given a source vertex s, explores the edges of G to discover every vertex that is reachable from s
  - Compute the distance (smallest number of edges) from s to each reachable vertex
  - Produce a breadth-first tree with root s that contains all reachable vertices
  - Compute the shortest path from s to each reachable vertex
- BFS discovers all vertices at distance k from s before discovering any vertices at distance k+1
Simple BFS from \( n \)

\[
\text{enqueue}(Q, n)
\]

\while \ Q \ \text{not empty}
\quad u = \text{dequeue}(Q)
\quad \text{process } u
\quad \text{for each } v \ \text{in Adjacency}(u) \ // \text{discover neighbours}
\quad \text{if } v \ \text{not yet discovered}
\quad \text{then enqueue}(Q, v)

Data Structure for BFS

- Adjacency list
- \( \text{color}[u] \) for each vertex
  - WHITE if \( u \) has not been discovered
  - BLACK if \( u \) and all its adjacent vertices have been discovered
  - GRAY if \( u \) has been discovered, but has some adjacent white vertices
- \( \text{FronMer} \) for the discovery and undiscovered vertices
- \( \text{d}[u] \) for the distance from (source) \( s \) to \( u \)
- \( \pi[u] \) for predecessor of \( u \)
- \( \text{FIFO queue } Q \) to manage the set of gray vertices
  - \( Q \) stores all the gray vertices

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**Example (BFS)**

(Courtesy of Prof. Jim Anderson)

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**Example (BFS)**
Example (BFS)

```
Q: r x 1 2 2
```

Example (BFS)

```
Q: l x 2 2 2
```

Example (BFS)

```
Q: x w 2 3 3
```

Example (BFS)

```
Q: u y 3 3
```

Example (BFS)

```
Q: y 3
```
Example (BFS)

```
Q: ∅
```

Example (BFS) – BF Tree

```
BF Tree
```

Analysis of BFS

- $O(|V|+|E|) = O(n+m)$
  - Each vertex is en-queued (O(1)) at most once $\Rightarrow O(n)$
  - No vertex is re-painted white
    - Vertex is inserted into queue and retrieved from there only once
    - Each adjacency list is scanned at most once $\Rightarrow O(m)$

Shortest path

- Print out the vertices on a shortest path from $s$ to $v$

```
PRINT-PATH(\(G, s, v\))
1 if \(v = s\)
2 then print \(s\)
3 else if \(\pi[v] = \text{NIL}\)
4 then print "no path from" \(s\) "to" \(v\) "exists"
5 else PRINT-PATH(\(G, s, \pi[v]\))
6 print \(v\)
```

PRINT-PATH Illustration

```
Output: s w t u
```
Connected Components

The connected components of an undirected graph are the separate “pieces” of the graph such that there is no connection between the pieces. Many seemingly complicated problems reduce to finding or counting connected components. For example, testing whether a puzzle such as Rubik’s cube or the 15-puzzle can be solved from any position is really asking whether the graph of legal configurations is connected. Anything we discover during a BFS must be part of the same connected component. We then repeat the search from any undiscovered vertex (if one exists) to define the next component, until all vertices have been found.

Bipartite graphs

- people and groups
- men-women
- Stable marriage
  - find matching that will not be “broken” by inevitable divorces
- Apples and Oranges

Two-Coloring Graphs

The vertex coloring problem seeks to assign a label (or color) to each vertex of a graph such that no edge links any two vertices of the same color. A graph is bipartite if it can be colored without conflicts while using only two colors. Bipartite graphs are important because they arise naturally in many applications. For example, consider the “bad-sex-with” graph in a heterosexual world. Men have sex only with women, and vice versa. Thus gender defines a legal two-coloring.

Finding a Two-Coloring

We can augment breadth-first search so that whenever we discover a new vertex, we color it the opposite of its parent.

def color(graph):
    for (v, u) in graph:
        if (color[v] == color[u]):
            print("No solution!")
            bipartite = False
        color[u] = complement(color[v])
        color[v] = complement(color[v])
    return True

We can assign the first vertex in any connected component to be whatever color/sex we wish.

Problem of the Day

Prove that in a breadth-first search on a undirected graph $G$, every edge in $G$ is either a tree edge or a cross edge, where a cross edge $(x, y)$ is an edge where $x$ is neither an ancestor or descendant of $y$. 
Depth-First Search (DFS)

- DFS: search deeper in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex \( v \) that still has unexplored edges leaving it
  - When all of \( v \)'s edges have been explored (finished), the search backtracks to explore edges leaving the vertex from which \( v \) was discovered
  - This process continues until we have discovered all the vertices that are reachable from the original source vertex
  - If any undiscovered vertices remain, then one of them is selected as a new source and the search is repeated from that source
- DFS will create a forest of DFS-trees

Simple DFS from \( n \)

```
push ( Q, n )
while u = pop (Q)
    process u
    for each v in reverse Adjacency( u )
        push ( Q, v )
```

Recursive:

```
DFS( u ) :
    process u
    for each v in Adjacency( u )
        DFS( v ) if v undiscovered
```

Data Structure for DFS

- Adjacency list
- color[\( u \)] for each vertex
  - WHITE if \( u \) has not been discovered
  - GRAY if \( u \) is discovered but not finished
  - BLACK if \( u \) is finished
- Timestamps: \( 1 \leq d[\( u \)] < f[\( u \)] \leq 2|V| \)
  - \( d[\( u \)] \) records when \( u \) is first discovered (and grayed)
  - \( f[\( u \)] \) records when the search finishes examining \( u \)'s adjacency list (and blacken \( u \))
- \( \pi[\( u \)] \) for predecessor of \( u \)

The Key Idea with DFS

A depth-first search of a graph organizes the edges of the graph in a precise way.
In a DFS of an undirected graph, we assign a direction to each edge, from the vertex which discover it:

```
DFS( \( G \) )
1 for each vertex \( u \in V[G] \)
2    do color[\( u \)] <- WHITE
3        \( \pi[\( u \)] \leftarrow \text{NIL} \)
4    time <- 0
5 for each vertex \( u \in V[G] \)
6    do if color[\( u \)] = WHITE
7        then DFS-Visit(\( u \))
```
DFS-visit — visit all reachable nodes

DFS-Visit(u)
1. color[u] ← GRAY  \(\triangleright\) White vertex \(u\) has just been discovered.
2. time ← time + 1
3. \(d[u] ← time\)
4. for each \(v \in A_d[u]\) \(\triangleright\) Explore edge \((u, v)\).
5. do if color[v] = WHITE
6. \hspace{1cm} then \(\pi[v] ← u\)
7. \hspace{1cm} DFS-Visit(v)
8. color[u] ← BLACK \(\triangleright\) Blacken \(u\); it is finished.
9. \(f[u] ← time ← time + 1\)

Example (DFS) (Courtesy of Prof. Jim Anderson)
Example (DFS)

Example (DFS)

Example (DFS)

Example (DFS)

Example (DFS)
Properties of DFS

- Time complexity: $\Theta(V+E)$
  - Loops on lines 1-3 and 5-7 of DFS: $\Theta(V)$
  - DFS-Visit
    - Called exactly once for each vertex
    - Loops on lines 4-7 for a vertex $v$: $|\text{Adj}[v]|$
    - Total time $= \sum_{v \in V} |\text{Adj}[v]| = \Theta(E)$

- DFS results in a forest of trees
- Discovery and finishing times have parenthesis structure

Depth-First Search

DFS has a neat recursive implementation which eliminates the need to explicitly use a stack. Discovery and final times are a convenience to maintain.

```c
if (discovery[y] == FALSE) {
    parent[y] = v;
    process(v, y);
    discovery[y] = time;
    time += 1;
}
```

Another Example of DFS

```
Another Example of DFS
```

DFS
- Stack

BFS
- Queue
Randomised search
Use: Randomized Queue

Parenthesis theorem

In any depth-first search of a (directed or undirected) graph \( G = (V, E) \), for any two vertices \( u \) and \( v \), exactly one of the following three conditions holds:

1. the intervals \([d(u), f(u)]\) and \([d(v), f(v)]\) are entirely disjoint, and neither \( u \) nor \( v \) is a descendant of the other in the depth-first forest,
2. the interval \([d(u), f(u)]\) is contained entirely within the interval \([d(v), f(v)]\), and \( u \) is a descendant of \( v \) in a depth-first tree, or
3. the interval \([d(v), f(v)]\) is contained entirely within the interval \([d(u), f(u)]\), and \( v \) is a descendant of \( u \) in a depth-first tree.

Theorem 22.7
For all \( u, v \), exactly one of the following holds:
1. \( d(u) < f(u) < d(v) < f(v) \) or \( d(u) < f(u) < d(v) < f(v) < d(u) < f(u) \) and neither \( u \) nor \( v \) is a descendant of the other.
2. \( d(u) < d(v) < f(u) < f(v) \) and \( v \) is a descendant of \( u \).
3. \( d(v) < d(u) < f(u) < f(v) \) and \( u \) is a descendant of \( v \).

Corollary
For all \( u, v \), exactly one of the following holds:
- \( d(u) < d(v) < f(u) < f(v) \)
- \( d(u) < f(u) < d(v) < f(v) < d(u) < f(u) \) and neither \( u \) nor \( v \) is a descendant of the other.
- \( d(u) < d(v) < f(u) < f(v) \) and \( v \) is a descendant of \( u \).
- \( d(v) < d(u) < f(u) < f(v) \) and \( u \) is a descendant of \( v \).
**Comp 122, Fall 2004**

**Depth-First Trees**
- Predecessor subgraph defined slightly different from that of BFS.
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{ (\pi[v], v) : v \in V$ and $\pi[v] \neq \text{NIL} \}$.  
  - How does it differ from that of BFS?
  - The predecessor subgraph $G_\pi$ forms a **depth-first forest** composed of several **depth-first trees**. The edges in $E_\pi$ are called **tree edges**.

**Example (Parenthesis Theorem)**

$$\begin{align*}
( 2 & ( y ( x x ) y ) ( w ( w ) x ) ) & ( v ( v ) ( u u ) ) \\
\end{align*}$$

**Classification of Edges**
- **Tree edge**: in the depth-first forest. Found by exploring $(u, v)$. -- $v$ was white
- **Back edge**: $(u, v)$, where $u$ is a descendant of $v$ (in the depth-first tree). -- $v$ was gray
- **Forward edge**: $(u, v)$, where $v$ is a descendant of $u$, but not a tree edge. -- $v$ was black and $d[u] < d[v]$
- **Cross edge**: any other edge. Can go between vertices in same depth-first tree or in different depth-first trees. -- $v$ was black and $d[u] > d[v]$

**Types of edges in DFS**

- **Tee edge**
- **Back edge**
- **Forward edge**
- **Cross edge**

**White-path Theorem**

*Theorem 22.9*  
$v$ is a descendant of $u$ if and only if at time $d[u]$, there is a path $u \cdots v$ consisting of only white vertices. (Except for $u$, which was just colored gray.)

**Proof**

We begin with the case in which $d[u] < d[v]$.  
- There are two subcases to consider, according to whether $d[v] < f[u]$ or not.
- The first subcase occurs when $d[v] < f[u]$, so $v$ was discovered while $u$ was still gray. This implies that $v$ is a descendant of $u$. Moreover, since $v$ was discovered more recently than $u$, all of its outgoing edges are explored, and $v$ is finished, before the search returns to and finishes $u$. In this case, therefore, the interval $[d[v], f[v]]$ is entirely contained within the interval $[d[u], f[u]]$.
- In the other subcase, $f[u] < d[v]$, and inequality (22.2) implies that the intervals $[d[u], f[u]]$ and $[d[v], f[v]]$ are disjoint.
- Because the intervals are disjoint, neither vertex was discovered while the other was gray, and so neither vertex is a descendant of the other.
- The case in which $d[v] < d[u]$ is similar, with the roles of $u$ and $v$ reversed in the above argument.

**Definition:**  
Forest: An acyclic graph $G$ that may be disconnected.
Edge Classification for DFS

Every edge is either:

1. A Tree Edge
2. A Back Edge
3. A Forward Edge
4. A Cross Edge

On any particular DFS or BFS of a directed or undirected graph, each edge gets classified as one of the above.

DFS: Tree Edges and Back Edges Only

The reason DFS is so important is that it defines a very nice ordering to the edges of the graph.

In a DFS of an undirected graph, every edge is either a tree edge or a back edge.

Why? Suppose we have a forward edge. We would have encountered \((v, u)\) when expanding \(v\), so this is a back edge.

No Cross Edges in DFS

Suppose we have a cross-edge

DFS Application: Finding Cycles

Back edges are the key to finding a cycle in an undirected graph.

Any back edge going from \(x\) to an ancestor \(y\) creates a cycle with the path in the tree from \(y\) to \(x\).

DFS-visit – visit all reachable nodes

Types of edges in DFS

\[\text{DFS-Visit}(v)
\]

1. \(\text{color}[u] \leftarrow \text{GRAY}\) ▷ White vertex \(u\) has just been discovered.
2. \(\text{time} \leftarrow \text{time} + 1\)
3. \(d[u] \leftarrow \text{time}\)
4. for each \(v \in \text{Adj}[u]\) ▷ Explore edge \((u, v)\).
5. do if \(\text{color}[v] = \text{WHITE}\)
6. then \(\pi[v] \leftarrow u\)
7. \(\text{DFS-Visit}(v)\)
8. \(\text{color}[v] \leftarrow \text{BLACK}\) ▷ Blacken \(v\); it is finished.
9. \(f[u] \leftarrow \text{time} \leftarrow \text{time} + 1\)
Lemma — DAG acyclicity

- DAG is acyclic if and only if DFS of G yields no back edges
  - Suppose that there is a back edge \((u, v)\). Then vertex \(v\) is an ancestor of vertex \(u\) in the depth-first forest. Thus a path from \(v\) to \(u\) in \(G\), and the back edge \((u, v)\) completes a cycle.
  - Suppose that \(G\) contains a cycle \(c\). We show that a DFS of \(G\) yields a back edge. Let \(u\) be the first vertex to be discovered in \(c\), and let \((u, v)\) be the preceding edge in \(c\). At time \(d[u]\), the vertices of \(c\) form a path of white vertices from \(v\) to \(u\). By the white-path theorem (Theorem 22.9), vertex \(u\) becomes a descendant of \(v\) in the depth-first forest. Therefore, \((u, v)\) is a back edge.

Articulation Vertices

Suppose you are a terrorist, seeking to disrupt the telephone network. Which station do you blow up?

An articulation vertex is a vertex of a connected graph whose deletion disconnects the graph. Clearly connectivity is an important concern in the design of any network. Articulation vertices can be found in \(O(n(m+n))\) — just delete each vertex to do a DFS on the remaining graph to see if it is connected.

A Faster \(O(n+m)\) DFS Algorithm

In a DFS tree, a vertex \(v\) (other than the root) is an articulation vertex iff \(v\) is not a leaf and some subtree of \(v\) has no back edge incident until a proper ancestor of \(v\).

Topological Sorting

A directed, acyclic graph has no directed cycles.

A topological sort of a graph is an ordering on the vertices so that all edges go from left to right. DAGs (and only DAGs) has at least one topological sort (here \(G, A, B, C, F, E, D\)).
Topological Sort

- A topological sort of a directed acyclic graph (DAG) is a linear order of all its vertices such that if $G$ contains an edge $(u, v)$, then $u$ appears before $v$ in the ordering.
  - If the graph contains cycles, no linear ordering is possible.
  - A topological sort can be viewed as an ordering of its vertices along a horizontal line so that all directed edges go from left to right.
- DAG are used in many applications to indicate precedence among events.

Example
(Courtesy of Prof. Jim Anderson)

Example
(Courtesy of Prof. Jim Anderson)
Example

Linked List:

A – B – D

Example

Linked List:

A – B – D

Example

Linked List:

A – B – D

Example

Linked List:

A – B – D

Example

Linked List:

A – B – D

Example

Linked List:

A – B – D
Example Linked List:

A
B
D
C
E
1/4
2/3
5/8
6/7
9/10

Example Linked List:

A
B
D
C
E
1/4
2/3
5/8
6/7
9/10

Theorem: Correctness of Topological sort

- TOPOLOGICAL-SORT(G) produces a topological sort of a directed acyclic graph G
  - Suppose that DFS is run on a given DAG G to determine finishing times for its vertices. It suffices to show that for any pair of distinct vertices u, v, if there is an edge in G from u to v, then \( f(v) < f(u) \).
  - The linear ordering is corresponding to finishing time ordering
  - Consider any edge \((u, v)\) explored by DFS(G). When this edge is explored, \( v \) cannot be gray (otherwise, \((u, v)\) will be a back edge). Therefore, \( v \) must be either white or black
    - If \( v \) is white, \( v \) becomes a descendant of \( u \), \( f(v) < f(u) \) (ex. pants & shoes)
    - If \( v \) is black, it has already been finished, so that \( f(v) \) has already been set \( f(v) < f(u) \) (ex. belt & jacket)

Strongly Connected Components

A directed graph is strongly connected iff there is a directed path between any two vertices. The strongly connected components of a graph is a partition of the vertices into subsets (maximal) such that each subset is strongly connected.

Observe that no vertex can be in two maximal components, so it is a partition.
Strongly Connected Components

- $G$ is strongly connected if every pair $(u, v)$ of vertices in $G$ is reachable from one another.
- A strongly connected component (SCC) of $G$ is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, both $u \sim v$ and $v \sim u$ exist.

Component Graph

- $G^{SCC} = (V^{SCC}, E^{SCC})$.
- $V^{SCC}$ has one vertex for each SCC in $G$.
- $E^{SCC}$ has an edge if there’s an edge between the corresponding SCC’s in $G$.
- $G^{SCC}$ for the example considered:

$G^{SCC}$ is a DAG

Lemma 22.13
Let $C$ and $C'$ be distinct SCC’s in $G$, let $u, v \in C, u', v' \in C'$, and suppose there is a path $u \sim \cdots \sim v$ in $G$. Then there cannot also be a path $v' \sim \cdots \sim u'$ in $G$.

Proof:
- Suppose there is a path $v' \sim \cdots \sim v$ in $G$.
- Then there are paths $u \sim \cdots \sim v'$ and $v' \sim \cdots \sim u$ in $G$.
- Therefore, $u$ and $v'$ are reachable from each other, so they are not in separate SCC’s.

Transpose of a Directed Graph

- $G^T = \text{transpose}$ of directed $G$.
  - $G^T = (V, E^T), E^T = \{(u, v) : (v, u) \in E\}$.
  - $G^T$ is $G$ with all edges reversed.
- Can create $G^T$ in $O(V + E)$ time if using adjacency lists.
- $G$ and $G^T$ have the same SCC’s. ($u$ and $v$ are reachable from each other in $G$ if and only if reachable from each other in $G^T$. )
Algorithm to determine SCCs

Let $G$ be a directed graph.

1. call DFS($G$) to compute finishing times $f[u]$ for all $u$
2. compute $G^T$
3. call DFS($G^T$), but in the main loop, consider vertices in order of decreasing $f[u]$ (as computed in first DFS)
4. output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

**Time:** $O(V + E)$.

**Example:** On board.

Kosaraju’s algorithm. 1978 ?
Tarjan – 1972
Gabow – 1990 (Chariyan, Melhorn 1996)

---

How does it work?

- **Idea:**
  - By considering vertices in second DFS in decreasing order of finishing times from first DFS, we are visiting vertices of the component graph in topologically sorted order.
  - Because we are running DFS on $G^T$, we will not be visiting any vertex $v$ from a set $U$ where $v$ and $u$ are in different components.
- **Notation:**
  - $d[u]$ and $f[u]$ always refer to the first DFS.
  - Extend notation for $d$ and $f$ to sets of vertices $U \subseteq V$:
    - $d(U) = \min_{u \in U} d[u]$ (earliest discovery time)
    - $f(U) = \max_{u \in U} f[u]$ (latest finishing time)

---

SCCs and DFS finishing times

**Lemma 22.14**
Let $C$ and $C'$ be distinct SCC's in a graph $G = (V, E)$. Suppose there exists an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

**Proof:**
- **Case 1:** $d(C) < d(C')$
  - Let $s$ be the first vertex discovered in $C$.
  - At time $d(s)$, all vertices in $C$ are white. Thus, there exist paths of white vertices from $s$ to all vertices in $C$ and $C'$.
  - By the white-path theorem, all vertices in $C$ and $C'$ are descendants of $s$ in depth-first tree.
  - By the parenthesis theorem, $f(C) = f(C') > f(C')$.

---
SCCs and DFS finishing times

**Lemma 22.14**
Let \( C \) and \( C' \) be distinct SCC’s in \( G = (V, E) \). Suppose there is an edge \( (u, v) \in E \) such that \( u \in C \) and \( v \in C' \). \( f(C) > f(C') \).

**Proof:**
- Case 2: \( d(C) > d(C') \)
  - Let \( y \) be the first vertex discovered in \( C' \).
  - At time \( d[y] \), all vertices in \( C' \) are white and there is a white path from \( y \) to each vertex in \( C' \) (since all vertices in \( C' \) are white and \( y \) is discovered).
  - By the earlier lemma, since there is an edge \( (u, v) \), we cannot have a path from \( C' \) to \( C \).
  - No vertex in \( C \) is reachable from \( y \).

  - Therefore, at time \( f[y] \), all vertices in \( C \) are still white.
  - Therefore, for all \( w \in C \), \( f[w] > f[y] \), which implies that \( f(C) > f(C') \).

**Corollary 22.15**
Let \( C \) and \( C' \) be distinct SCC’s in \( G = (V, E) \). Suppose there is an edge \( (u, v) \in E \), where \( u \in C \) and \( v \in C' \). Then \( f(C) < f(C') \).

**Proof:**
- \( (u, v) \in E \) \( \implies \ (v, u) \in E \).
- Since SCC’s of \( G \) and \( G^T \) are the same, \( f(C') > f(C) \), by Lemma 22.14.

**Correctness of SCC**
- When we do the second DFS, on \( G^T \), start with SCC \( C \) such that \( f(C) \) is maximum.
  - The second DFS starts from some \( x \in C \), and it visits all vertices in \( C \).
  - Corollary 22.15 says that since \( f(C) > f(C') \) for all \( C \neq C' \), there are no edges from \( C \) to \( C' \) in \( G^T \).
  - Therefore, DFS will visit only vertices in \( C \).
  - Which means that the depth-first tree rooted at \( x \) contains exactly the vertices of \( C \).

**Strongly Connected Components Example**

**Why does strongly connected component method work?**
- Seel CLRS (2-3 pages)
Advanced Algorithmics (6EAP)
Graphs II
Jaak Vilo
2011 Spring

WEIGHTED GRAPH ALGORITHMS

Weighted Graph Algorithms
Beyond DFS/BFS exists an alternate universe of algorithms for edge-weighted graphs. Our adjacency list representation quietly supported these graphs:
def struct {
    int y;
    int weight;
    struct edgenode *next;
} edgenode;

Minimum Spanning Tree
Definition: Given an undirected graph, and for each edge \((v, u) \in E\), we have a weight \(w(u, v)\) specifying the cost to connect \(u\) and \(v\). Find an acyclic subset \(T \subseteq E\) that connects all of the vertices and whose total weight is minimized
- \(w(T) = \sum_{(u,v) \in T} w(u, v)\)
- May have more than one MST with the same weight
- Two classic algorithms: \(O(E \log V)\)Greedy Algorithms
  - Kruskal’s algorithm
  - Prim’s algorithm

Minimum Spanning Trees
A tree is a connected graph with no cycles. A spanning tree is a subgraph of \(G\) which has the same set of vertices of \(G\) and is a tree.
A minimum spanning tree of a weighted graph \(G\) is the spanning tree of \(G\) whose edges sum to minimum weight. There can be more than one minimum spanning tree in a graph — consider a graph with identical weight edges.

Equal weights in left fully connected graph (a)
Why Minimum Spanning Trees?

The minimum spanning tree problem has a long history – the first algorithm dates back at least to 1926.
Minimum spanning tree is always taught in algorithm courses since (1) it arises in many applications, (2) it is an important example where greedy algorithms always give the optimal answer, and (3) Clever data structures are necessary to make it work.
In greedy algorithms, we make the decision of what next to do by selecting the best local option from all available choices – without regard to the global structure.

Applications of Minimum Spanning Trees

Minimum spanning trees are useful in constructing networks, by describing the way to connect a set of sites using the smallest total amount of wire.
Minimum spanning trees provide a reasonable way for clustering points in space into natural groups.
What are natural clusters in the friendship graph?

Minimum Spanning Trees and TSP

When the cities are points in the Euclidean plane, the minimum spanning tree provides a good heuristic for traveling salesman problems.
The optimum traveling salesman tour is at most twice the length of the minimum spanning tree.

Fully connected graph.
Find a MST?

MST

MST-approximation of TSP

Images: http://www.personal.kent.edu/~rmuhamma/
Growing a Minimum Spanning Tree (MST)

- Generic algorithm
  - Grow MST one edge at a time
  - Manage a set of edges $A$, maintaining the following loop invariant:
    - Prior to each iteration, $A$ is a subset of some MST
    - At each iteration, we determine an edge $(u, v)$ that can be added to $A$ without violating this invariant
      - $A \cup \{(u, v)\}$ is also a subset of a MST
      - $(u, v)$ is called a safe edge for $A$

**How to Find a Safe Edge?**

- **Theorem.** Let $A$ be a subset of $E$ that is included in some MST, let $(S, V-S)$ be any cut of $G$ that respects $A$, and let $(u, v)$ be a light edge crossing $(S, V-S)$. Then edge $(u, v)$ is safe for $A$
  - Cut $(S, V-S)$: a partition of $V$
  - Crossing edge: one endpoint in $S$ and the other in $V-S$
  - $A$ respects a set of $A$ of edges if no edges in $A$ crosses the cut
  - A light edge crossing a cut if its weight is the minimum of any edge crossing the cut

**Illustration of Theorem 23.1**

- $A=\{(a, b), (c, i), (h, d), (g, f)\}$
- $S=\{a, b, c, i, e\}$
- $V-S=\{h, g, f, d\}$
- many kinds of cuts satisfying the requirements of Theorem 23.1
- $(c, f)$ is the light edges crossing $S$ and $V-S$ and will be a safe edge

**Proof of Theorem 23.1**

- Let $T$ be a MST that includes $A$, and assume $T$ does not contain the light edge $(u, v)$, since if it does, we are done.
- Construct another MST $T'$ that includes $A \cup \{(u, v)\}$ from $T$
  - Next slide
  - $T'=T-(v,x)) \cup \{(u,v)\}$
  - $T'$ is also a MST since $W(T')=W(T)+w(x,v)+w(u,v)$
- $(u, v)$ is actually a safe edge for $A$
  - Since $A \subset T$ and $(x, v) \notin A \Rightarrow A \subset T'$
  - $A \cup \{(u, v)\} \subset T'$
Proper use of GENERIC-MST

As the algorithm proceeds, the set $A$ is always acyclic.

$G_U(V, A)$ is a forest, and each of the connected components of $G_A$ is a tree.

Any safe edge $(u, v)$ for $A$ connects distinct component of $G_u$, since $A \cup \{(u, v)\}$ must be acyclic.

Corollary 23.2. Let $A$ be a subset of $E$ that is included in some MST, and let $C = (V_C, E_C)$ be a connected components (tree) in the forest $G_U(V, A)$. If $(u, v)$ is a light edge connecting $C$ to some other components in $G_U$, then $(u, v)$ is safe for $A$.

The Algorithms of Kruskal and Prim

- Kruskal’s Algorithm
  - $A$ is a forest
  - The safe edge added to $A$ is always a least-weight edge in the graph that connects two distinct components.

- Prim’s Algorithm
  - $A$ forms a single tree
  - The safe edge added to $A$ is always a least-weight edge connecting the tree to a vertex not in the tree.

Prim’s Algorithm

If $G$ is connected, every vertex will appear in the minimum spanning tree. If not, we can talk about a minimum spanning forest.

Prim’s algorithm starts from one vertex and grows the rest of the tree at a time.

As a greedy algorithm, which edge should we pick? The cheapest edge with which can grow the tree by one vertex without creating a cycle.

Prim’s Algorithm (Pseudocode)

During execution each vertex $v$ is either in the tree, \textit{fringe} (meaning there exists an edge from a tree vertex to $v$) or \textit{unseen} (meaning $v$ is more than one edge away).

Prim-MST($G$)

Select an arbitrary vertex $s$ to start the tree from.
While (there are still non-tree vertices)
  Select the edge of minimum weight between a tree and \textit{fringe} and make

This creates a spanning tree, since no cycle can be introduced, but is it minimum?
**Key idea of Prim’s algorithm**

Select a vertex to be a tree-node

while (there are non-tree vertices)

  if (there is no edge connecting a tree node with a non-tree node)
    return “no spanning tree”

  select an edge of minimum weight between a tree node and a non-tree node

  add the selected edge and its new vertex to the tree

return tree

---

**Prim’s Algorithm in Action**

![Diagram of Prim’s Algorithm in Action]

---

**Prim’s Algorithm (Cont.)**

- How to efficiently select the safe edge to be added to the tree?
  - Use a min-priority queue Q that stores all vertices not in the tree
  - Based on key[v], the minimum weight of any edge connecting v to a vertex in the tree
    - Key[v] = ∞ if no such edge
  - π[v] = parent of v in the tree
  - A = {(v, π[v]) : v ∈ V - {r} - Q} → finally Q = empty

---

**Prim’s Algorithm**

1. for each u ∈ V
2.   do D[u] ← ∞
3. D[r] ← 0
4. MH ← make-heap(D, {} //No edges
5. T ← ∅
6. while MH ≠ ∅ do
7.     (u, e) ← MH.extractMin()
8.     add (u, e) to T
9.     for each v ∈ Adjacent (u)
10.    do if v ∈ MH & w(u, v) < D[v]
11.       then D[v] ← w(u, v)
12.       MH.decreaseDistance(D[v], v, (u, v))
13. return T //T is a MST

---

**MST-PRIM(G, w, r)**

1. for each u ∈ V[G]
2.   do key[u] ← ∞
3. π[u] ← NIL
4. key[r] ← 0
5. Q ← V[G]
6. while Q ≠ ∅ do
7.     u ← EXTRACT-MIN(Q)
8.     for each v ∈ Adjacent[u]
9.       do if v ∈ Q and w(u, v) < key[v]
10.      then π[v] ← u
11.      key[v] ← w(u, v)

---

**Lines 1-5 initialize the min-heap (MH) to contain all vertices.**

**Distance for all vertices, except r, are set to infinity.**

**r is the starting vertex of the T**

**The T so far is empty**
Illustration of MST-PRIM

Properties of MST-PRIM

- Prior to each iteration of the while loop of lines 6—11
  - \( A = \{(v, \pi[v]): v \in V-(f)-Q\} \)
  - The vertices already placed into the MST are those in \( V-Q \)
  - For all vertices \( v \in Q \), if \( \pi[v] \neq \text{NIL} \), then key\( [v] \) = \( \infty \) and key \( [v] \) is the weight of a light edge \( (v, \pi[v]) \) connecting \( v \) to some vertex already placed into the MST
- Line 7: identify a vertex \( u \in Q \) incident on a light edge crossing \( (V-Q, Q) \) \( \rightarrow \) add \( u \) to \( V-Q \) and \((u, \pi[u])\) to \( A \)
- Lines 8—11: update key and \( \pi \) of every vertex \( v \) adjacent to \( u \) but not in the tree

Performance of MST-PRIM

- Use binary min-heap to implement the min-priority queue \( Q \)
  - BUILD-MIN-HEAP (line 5): \( O(V) \)
  - The body of while loop is executed \( |V| \) times
    - EXTRACT-MIN: \( O(lg V) \)
  - The for loop in lines 8-11 is executed \( O(E) \) times altogether
    - Line 11: DECREASE-KEY operation: \( O(lg V) \)
  - Total performance = \( O(V \ lg V + E \ lg V) = O(E \ lg V) \)
- Use Fibonacci heap to implement the min-priority queue \( Q \)
  - \( O(E + V \ lg V) \)

Why is Prim Correct?

We use a proof by contradiction:
Suppose Prim’s algorithm does not always give the minimum cost spanning tree on some graph.
If so, there is a graph on which it fails.
And if so, there must be a first edge \((x,y)\) Prim adds such that the partial tree \( V' \) cannot be extended into a minimum spanning tree.

Kruskal’s Algorithm

Since an easy lower bound argument shows that every edge must be looked at to find the minimum spanning tree, and the number of edges \( m = O(n^2) \), Prim’s algorithm is optimal in the worst case. Is that all she wrote?
The complexity of Prim’s algorithm is independent of the number of edges. Can we do better with sparse graphs? Yes!
Kruskal’s algorithm is also greedy. It repeatedly adds the smallest edge to the spanning tree that does not create a cycle.
Why is Kruskal’s algorithm correct?

Again, we use proof by contradiction.
Suppose Kruskal’s algorithm does not always give the
minimum cost spanning tree on some graph.
If so, there is a graph on which it fails.
And if so, there must be a first edge \((x, y)\) Kruskal adds such
that the set of edges cannot be extended into a minimum
spanning tree.
When we added \((x, y)\) there previously was no path between
\(x\) and \(y\), or it would have created a cycle
Thus if we add \((x, y)\) to the optimal tree it must create a cycle.
At least one edge in this cycle must have been added after
\((x, y)\), so it must have a heavier weight.
Deleting this heavy edge leave a better MST than the optimal
tree? A contradiction!

How fast is Kruskal’s algorithm?

What is the simplest implementation?
• Sort the \(m\) edges in \(O(m \log m)\) time.
• For each edge in order, test whether it creates a cycle the
  forest we have thus far built — if so discard, else add to
  forest. With a BFS/DFS, this can be done in \(O(n)\) time
  (since the tree has at most \(n\) edges).
The total time is \(O(mn)\), but can we do better?

Fast Component Tests Give Fast MST

Kruskal’s algorithm builds up connected components. Any
edge where both vertices are in the same connected com-
ponent create a cycle. Thus if we can maintain which vertices
are in which component fast, we do not have test for cycles!
• Same component\((v_1, v_2)\) — Do vertices \(v_1\) and \(v_2\) lie in
  the same connected component of the current graph?
• Merge components\((C_1, C_2)\) — Merge the given pair of
  connected components into one component.

Fast Kruskal Implementation

Put the edges in a heap
\(\text{count} = 0\)
while \((\text{count} < n - 1)\) do
  get next edge \((v, w)\)
  if \(\text{component}(v) \neq \text{component}(w)\)
    add to \(T\)
    \(\text{component}(v) = \text{component}(w)\)
If we can test components in \(O(\log n)\), we can find the MST
in \(O(m \log m)\)!
Question: Is \(O(m \log n)\) better than \(O(m \log m)\)?

Union-Find Programs

We need a data structure for maintaining sets which can test
if two elements are in the same set and merge two sets together.
These can be implemented by \textbf{union} and \textbf{find} operations, where
• \textbf{Find}(\(i\)) — Return the label of the root of tree containing
  element \(i\), by walking up the parent pointers until there is
  no where to go.
• \textbf{Union}(\(j\)) — Link the root of one of the trees (say
  containing \(i\)) to the root of the tree containing the other
  (say \(j\)) so find\((i)\) now equals find\((j)\).

This path compression will let us do better than \(O(n \log n)\)
for \(n\) union-finds.
\(O(n)\)? Not quite. . . Difficult analysis shows that it takes
\(O(n \alpha(n))\) time, where \(\alpha(n)\) is the inverse Ackerman function
and \(\alpha(\text{number of atoms in the universe}) = 5\).
**Problem of the Day**

Suppose we are given the minimum spanning tree $T$ of a given graph $G$ (with $n$ vertices and $m$ edges) and a new edge $e = (u, v)$ of weight $w$ that we will add to $G$. Give an efficient algorithm to find the minimum spanning tree of the graph $G + e$. Your algorithm should run in $O(n)$ time to receive full credit, although slower but correct algorithms will receive partial credit.

---

**Table 25.1 Cost of MST algorithms**

This table summarizes the cost (worst-case running time) of various MST algorithms considered in this chapter. The formulas are based on the assumptions that an MST exists (which implies that $K$ is no smaller than $V - 1$) and that there are $X$ edges not longer than the longest edge in the MST (see Property 25.10). These worst-case bounds may be too conservative to be useful in predicting performance on real graphs. The algorithms run in near-linear time in a broad variety of practical situations.

<table>
<thead>
<tr>
<th>algorithm</th>
<th>worst-case cost</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prim (standard)</td>
<td>$V^2$</td>
<td>optimal for dense graphs</td>
</tr>
<tr>
<td>Prim (PFS, heap)</td>
<td>$E \log V$</td>
<td>conservative upper bound</td>
</tr>
<tr>
<td>Prim (PFS, U-heap)</td>
<td>$E \frac{V}{2}$</td>
<td>linear unless extremely sparse</td>
</tr>
<tr>
<td>Kruskal (partial sort)</td>
<td>$E \log V$</td>
<td>sort cost dominates</td>
</tr>
<tr>
<td>Boruvka</td>
<td>$E \log V$</td>
<td>conservative upper bound</td>
</tr>
</tbody>
</table>

---

**4-letter words, distance 1**

![Graph of 4-letter words, distance 1](LEDAtutorial)
Shortest paths between nodes in graph

- Practical applications
- Transportation
  - Cheapest or quickest way to travel from A to B
- Motion planning
  - Most natural way for a cartoon character to navigate between places
- Communications
  - Time to send a message; diameter of a graph,

Example: Predictive Mobile text Entry Messaging...

What was the message?

Weighting the Graph

The weight of each edge is a function of the probability that these two words will be next to each other in a sentence. ‘hive me’ would be less than ‘give me’, for example. The final system worked extremely well — identifying over 99% of characters correctly based on grammatical and statistical constraints.
Problem Definition

• Given a weighted, directed graph $G=(V, E)$ with weight function $w: E \rightarrow \mathbb{R}$. The weight of path $p=v_0, v_1, \ldots, v_k$ is the sum of the weights of its constituent edges:

$$w(p) = \sum_{i=1}^{k} w(e_{i-1,i})$$

• We define the shortest-path weight from $u$ to $v$ by

$$\delta(u,v) = \min\{ w(p) : u \rightarrow v \}$$

If there is a path from $u$ to $v$, otherwise.

• A shortest path from vertex $u$ to vertex $v$ is then defined as any path with $w(p) = \delta(u, v)$

Variants

• Single-source shortest paths problem – greedy

– Finds all the shortest paths of vertices reachable from a single source vertex $s$

• Single-destination shortest-path problem

– By reversing the direction of each edge in the graph, we can reduce this problem to a single-source problem

• Single-pair shortest-path problem

– No algorithm for this problem are known that run asymptotically faster than the best single-source algorithm in the worst case

• All-pairs shortest-path problem – dynamic programming

– Can be solved faster than running the single-source shortest-path problem for each vertex

Optimal Substructure of A Shortest-Path

• Lemma 24.1 (Subpath of shortest paths are shortest paths). Let $p=v_1, v_2, \ldots, v_k$ be a shortest path from vertex $v_i$ to $v_k$, and for any $i$ and $j$ such that $1 \leq i < j \leq k$, let $p_{ij} = v_1, v_2, \ldots, v_j$ be the subpath of $p$ from vertex $v_i$ to $v_j$.

Then $p_{ij}$ is a shortest path from vertex $v_i$ to $v_j$.

Negative-Weight Edges and Cycles

• Cannot contain a negative-weight cycle

• Of course, a shortest path cannot contain a positive-weight cycle

Examples of shortest paths depending on start node
RelaxaMon

For each vertex $v \in V$ of the graph, we maintain an attribute $d[v]$, which is an upper bound on the weight of a shortest path from source $s$ to $v$. We call $d[v]$ a shortest-path estimate.

**INITIALIZE-SINGLE-SOURCE** $(G, s)$
1. For each vertex $v \in V(G)$, do $d[v] \leftarrow \infty$
2. $\pi[v] \leftarrow \text{NIL}$
3. $d[s] \leftarrow 0$

Relaxation

Relaxing an edge $(u, v)$ consists of testing whether we can improve the shortest path found so far by going through $u$ and, if so, updating $d[v]$ and $\pi[v]$.

\[
\begin{align*}
\text{for each edge } (u, v) \in G.E & \text{ do } \\
& \text{ if } v.d > u.d + w(u, v) \\
& \quad \text{then } d[v] \leftarrow d[u] + w(u, v) \\
& \quad \pi[v] \leftarrow u
\end{align*}
\]

Bellman-Ford

Bellman-Ford $(G, w, s)$
1. Initialise-Single-Source$(G, S)$
2. for $i=1$ to $|G.V|-1$ /* $n-1$ rounds */ do
3. for each edge $(u, v) \in G.E$
4. RELAX$(u, v, w)$
5. for each edge $(u, v) \in G.E$
6. if $v.d > u.d + w(u, v)$
7. return FALSE
8. return TRUE

Bellman-Ford

- $O(V E)$
- Just repeatedly relax all edges.
  - Allow $V$ cycles to propagate through the network

Shortest paths on a DAG

1. DAG-Shortest-path$(G,w,s)$
2. topologically sort vertices
3. Initialise-single-source$(G, S)$
4. for each vertex $u$ in topological order
5. for each vertex $v \in G.Adj[u]$
6. RELAX$(u,v,w)$

$O(V + E)$
Dijkstra’s Algorithm

- Solve the single-source shortest-paths problem on a weighted, directed graph when all edge weights are nonnegative
- Data structure
  - S: a set of vertices whose final shortest-path weights have already been determined
  - Q: a min-priority queue keyed by their d values
- Idea
  - Repeatedly select the vertex \( u \in V-S \) (kept in Q) with the minimum shortest-path estimate, add \( s \) to \( S \), and relax all edges leaving \( u \)

Dijkstra’s Algorithm (Cont.)

\[
\text{DIJKSTRA}(G, w, s)
\]

1. \text{INITIALIZE-SINGLE-SOURCE}(G, s)
2. \( S \leftarrow \emptyset \)
3. \( Q \leftarrow V[G] \)
4. \textbf{while} \( Q \neq \emptyset \)
   \hspace{1em}5. \hspace{1em}\textbf{do} \( u \leftarrow \text{EXTRACT-MIN}(Q) \)
   \hspace{1em}6. \hspace{1em}\( S \leftarrow S \cup \{u\} \)
   \hspace{1em}7. \hspace{1em}\textbf{for} each vertex \( v \in \text{Adj}[u] \)
   \hspace{1.5em}8. \hspace{1.5em}\textbf{do} \( \text{RELAX}(u, v, w) \)

Note: relax requires updating of min values in Q.
Analysis of Dijkstra’s Algorithm

- Correctness: Theorem 24.6 (Loop invariant)
- Min-priority queue operations
  - INSERT (line 3)
  - EXTRACT-MIN (line 5)
  - DECREASE-KEY (line 8)
- Time analysis
  - Line 4-8: while loop $\rightarrow O(V)$
  - Line 7-8: for loop and relaxation $\rightarrow |E|$
  - Running time depends on how to implement min-priority queue
    - Simple array: $O(V^2 + E) = O(V^2)$
    - Binary min-heap: $O((V+E)\log V)$
    - Fibonacci min-heap: $O(V\log V + E)$

http://www.cs.utexas.edu/users/EWD/

- Edsger Wybe Dijkstra was one of the most influential members of computing science’s founding generation. Among the domains in which his scientific contributions are fundamental are
  - algorithm design
  - programming languages
  - program design
  - operating systems
  - distributed processing
  - formal specification and verification
  - design of mathematical arguments

Dijkstra’s Algo
1) Dijkstra is a Greedy based algorithm and similar to Prim’s MST algo.
2) Dijkstra doesn’t work for negative weight edges.
3) Time complexity of Dijkstra is $O(|E| + |V|\log |V|)$
4) Dijkstra’s algorithm is usually the working principle behind link-state routing protocols, OSPF and IS-IS

All pairs shortest paths

- Diameter of a graph (longest shortest path)
- Calculate the shortest path from each source
- Find the longest shortest path...
- Means to estimate/approximate it
Fast Fully Dynamic Landmark-based Estimation of Shortest Path Distances in Very Large Graphs

ABSTRACT

Computing the shortest path between a pair of vertices in a graph is a fundamental problem in graph algorithms. Existing methods fail in practice, rapidly reaching social networks with too many edges.

Keywords

Graph Databases, Social Networks, Landmark-based Estimation, Dynamic Updates.
Socially sensitive search

Naïve approach (Breadth-First-Search) requires 5-20 minutes

Landmark-based estimation

1 <= d <= 7

Basic Method

1 <= d <= 5

Landmark-based estimation

3 <= d <= 5

Mary Ann

Mary Lee

Basic Method

3 <= d <= 5

Mary Ann

Landmark-based estimation

3 <= d <= 5
Landmark-based estimation

Least common ancestor

Combining multiple landmarks

Shortest path tree

Least common ancestor

Combining multiple landmarks
Combining multiple landmarks

**Landmarks-BFS**

Given two nodes U and V:
1. Collect all paths from U and V to all landmarks
2. Run a BFS* on the induced subgraph

* or Dijkstra, or A*, or anything else

---

Landmark-based approximation

**Basic Method**

- LCA
- Shortcutting
- Landmarks-BFS

**Speed**

**Accuracy**

---

Insertion of an edge

---
Deletion – more complicated

| Dataset | $|V|$ | $|E|$ | $d$ | $\Delta$ | $S/|V|$ | $t_{pr}$ |
|---------|-----|-----|----|-------|-------|--------|
| DBLP    | 770K| 2.6M| 6.3| 23    | 80%   | 345 ms  |
| Orkut   | 3.1M| 117M| 5.7| 10    | 100%  | 8 sec   |
| Twitter | 41.7M| 1.2B| 4.2| 24    | 100%  | 9 min   |
| Skype   | 454M| 3.1B| 6.5| 59    | 85%   | 20 min  |

Evaluation - Data

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Graph file</th>
<th>Landmark file</th>
<th>Basic</th>
<th>LCA/SC/LBFS</th>
</tr>
</thead>
<tbody>
<tr>
<td>DBLP</td>
<td>27M</td>
<td>765K</td>
<td>10.0M</td>
<td>1.0M</td>
</tr>
<tr>
<td>Orkut</td>
<td>938M</td>
<td>30M</td>
<td>12.0M</td>
<td>1.0M</td>
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<td>40M</td>
<td>160M</td>
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<td>Skype</td>
<td>27G</td>
<td>433M</td>
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*per each landmark*
Timings: Query

<table>
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<th>Method</th>
<th>20</th>
<th>60</th>
<th>100</th>
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</thead>
<tbody>
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<td>0.56</td>
<td>0.91</td>
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<tr>
<td></td>
<td>LCA</td>
<td>1.06</td>
<td>2.43</td>
<td>3.69</td>
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<tr>
<td></td>
<td>SC</td>
<td>1.22</td>
<td>2.92</td>
<td>4.85</td>
</tr>
<tr>
<td></td>
<td>LBFS</td>
<td>5.10</td>
<td>13.24</td>
<td>16.25</td>
</tr>
</tbody>
</table>

Time for a batch of 500 queries / 500, in ms
- Linux, mmap, 32 cores, 256GB RAM

Timings: Updates

<table>
<thead>
<tr>
<th></th>
<th>DBLP</th>
<th>Orkut</th>
<th>Twitter</th>
<th>Skype</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Insertion</td>
<td>1μs</td>
<td>10μs</td>
<td>10μs</td>
</tr>
<tr>
<td></td>
<td>Deletion*</td>
<td>100μs</td>
<td>2ns</td>
<td>12ns</td>
</tr>
</tbody>
</table>

* very non-uniform

Outline

- Improvement to Basic Landmark method
- Dynamic updates
- Landmark selection
- Evaluation

Landmark selection method

- Landmark is good if it covers many shortest paths

Best Coverage

```
A  B
C  D
E  F
```

Best Coverage

```
A  X  Y  B
C  Y  X  D
E  Y  Z  F
```

...
27.3.2013

### Best Coverage

\[
\begin{align*}
A & \quad Y & \quad M & \quad B \\
C & \quad Y & \quad X & \quad D \\
E & \quad Y & \quad Z & \quad F
\end{align*}
\]

### Timings: Landmark selection

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Highest degree</th>
<th>Best coverage</th>
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</thead>
<tbody>
<tr>
<td>DBLP</td>
<td>140 ms</td>
<td>2 min</td>
</tr>
<tr>
<td>Orkut</td>
<td>2 s</td>
<td>15 min</td>
</tr>
<tr>
<td>Twitter</td>
<td>22 s</td>
<td>15 h</td>
</tr>
<tr>
<td>Skype</td>
<td>1 min</td>
<td>54 h</td>
</tr>
</tbody>
</table>

### Summary (Skype graph)

- **Network size**: 500M nodes, 3B edges
- **Landmark selection time (HD)**: 1 min / 54 hr
- **Landmark computation time**: 20 min x 100
- **Total space for 100 landmarks**: 170G
- **Avg query time (SC/LBFS)**: 5 ms / 16 ms
- **Avg edge insertion time**: 0.030 ms
- **Avg edge deletion time**: 11 ms
- **Avg relative error (SC/LBFS)**: 18% / 15%

### Questions

- **LCA**
- **Shortcutting**
- **Landmarks-BFS**
- **Dynamic updates**
- **Highest degree**
- **Best coverage**

- Precomp. 1+100x20 min
- Space 170G
- Query 5 ms / 16 ms
- Insertion 0.030 ms
- Deletion 11 ms
- Error 18% / 15%
Generalizations
- To weighted graph:
  - Use weighted shortest path trees
  - The dynamic update algorithm becomes slightly more complicated

- To directed graph:
  - Use two SPTs per landmark

Improvements
- Parallelization possible at most stages
- "Evolutionary" on-line selection of landmarks
- Use of landmark-based heuristics with A* for exact path possible (Goldberg et al., Ikeda et al.)

Timings: Query / Twitter

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Method</th>
<th>No. of Landmarks</th>
</tr>
</thead>
<tbody>
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<td>20</td>
</tr>
<tr>
<td>Twitter</td>
<td>Basic</td>
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<tr>
<td></td>
<td>LCA</td>
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<tr>
<td></td>
<td>SC</td>
<td>0.82</td>
</tr>
<tr>
<td></td>
<td>LBFS</td>
<td>240</td>
</tr>
</tbody>
</table>

Euclidean Networks
- In applications where networks model maps, our primary interest is often in finding the best route from one place to another. In this section, we examine a strategy for this problem: a fast algorithm for the source–sink shortest-path problem in Euclidean networks, which are networks whose vertices are points in the plane and whose edge weights are defined by the geometric distances between the points.
- These networks satisfy two important properties that do not necessarily hold for general edge weights. First, the distances satisfy the triangle inequality: The distance from s to d is never greater than the distance from s to x plus the distance from x to d. Second, vertex positions give a lower bound on path length: No path from s to d will be shorter than the distance from s to d. The algorithm for the source–sink shortest-paths problem that we examine in this section takes advantage of these two properties to improve performance.

Euclidean Path
We’ll look more closely at this with the A* algorithm (heuristic search)
Calculating paths by matrix operations

**Path of length 2**

\[ c_{ij} = \sum_k a_{ik} b_{kj} \]

The textbook algorithm for computing the product of two \(V\)-by-
\(V\) matrices computes, for each \(s\) and \(i\), the dot product of row \(s\) in the first matrix and row \(i\) in the second matrix, as follows:

\[
\text{for } (s = 0; s < V; s++) \\
\text{for } (t = 0; t < V; t++) \\
\text{for } (i = 0; C[s][t] = 0; i < V; i++) \\
C[s][t] += A[s][i] \cdot B[i][t];
\]

In matrix notation, we write this operation simply as \(C = A \cdot B\). This operation is defined for matrices comprising any type of entry for which \(0\), \(+\), and \(\cdot\) are defined. In particular, if the matrix entries are either `true` or `false` and we interpret `\&\&` to be the logical `and` operation and `\|\|` to be the logical `or` operation, then we have Boolean matrix multiplication. In Java, we can use the following version:

\[
\text{for } (s = 0; s < V; s++) \\
\text{for } (t = 0; t < V; t++) \\
\text{for } (i = 0; C[s][t] = false; i < V; i++) \\
\text{if } (A[s][i] \&\& B[i][t]) C[s][t] = true;
\]
Diagonal 1 = self-loop

\[
\begin{array}{c|ccccc}
\text{Diagonal 0 or 1} & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 1 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 \\
4 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[G^*G\]

Transitive closure

- Transitive closure of a digraph G is a graph G' with same vertices, and edge between any u and v from G if there is a path from u to v in G

Transitive closure

\[G \cdot G \cdot G \cdot ...\]

with opposite vertices and edge between any u and v from G if there is a path from u to v in G

Transitive closure

\[G_1[1][j] \Rightarrow G_1[1][k]\]

Exists link via j on diagonal - link to itself

Book: Sedgewick, Algorithms ...

- 19.3. Reachability and Transitive Closure
  - [proquest.safaribooksonline.com](http://proquest.safaribooksonline.com)
Complexity...

- for $i=1$ to $|V|$ do $V(i) = V(i-1) \cdot V$
- $V^3$ operations for $V^2$, $V^3$, ... $V^n$
- $\Rightarrow O(V^4)$

- Use exponential: $2 \Rightarrow 4 \Rightarrow 8 \Rightarrow 16$ ... steps.
- $V^2 \cdot V^2 = V^4$, $V^4 \cdot V^2 = V^6$, ... $\Rightarrow O(\log V) \cdot V^3$
- Can we avoid so many cycles?

Multiply:

for $(s = 0; s < V; s++)$
  for $(t = 0; t < V; t++)$
    for $(i = 0; C[s][t] = 0; i < V; i++)$
      $C[s][t] := A[s][i] \cdot B[i][t];$

Transitive closure:

for $(i = 0; i < V; i++)$
  for $(s = 0; s < V; s++)$
    for $(t = 0; t < V; t++)$
      if $(A[s][i] \&\& A[i][t])$ 
        $A[s][t] =$ true;

Property 19.3 With Warshall's algorithm, we can compute the transitive closure of a digraph in time proportional to $V^3$.

Proof: The running time is immediately evident from the structure of the code. We prove that it computes the transitive closure by induction on $i$. After the first iteration of the loop, the matrix has true in row $s$ and column $t$ and only if the digraph has either the edge $s \cdot t$ or the paths $s \cdot j \cdot t$. The second iteration checks all the paths between $s$ and $t$ that include $i$ and perhaps $0$, such as $s \cdot i \cdot t$, $s \cdot 0 \cdot i \cdot t$, and $s \cdot 0 \cdot i \cdot t$. We are led to the following inductive hypothesis: the $i$th iteration of the loop sets the bit in row $s$ and column $t$ in the matrix to true if and only if there is a directed path from $s$ to $t$ in the digraph that does not include any vertices with indices greater than $i$ (except possibly the endpoints $s$ and $t$). As just argued, the condition is true when $i = 0$, after the first iteration of the loop. Assuming that it is true for the $i$th iteration of the loop, there is a path from $s$ to $t$ that does not include any vertices with indices greater than $i$ and only if $(i)$ there is a path from $s$ to $t$ that does not include any vertex with indices greater than $i$ and only if $A[i][t] = true$, or $i$ is a path from $s$ to $t$, and $i$ is a path from $0$ to $i$, $i$ is a path from $i$ to $t$, and it is possible that there is a path from $s$ to $t$ in the digraph that does not include any vertex with indices greater than $i$, and the loop sets the bit in row $s$ and column $t$ as true (by hypothesis), so the inner loop sets $A[s][t] =$ true.

Proof: transitive closure by induction on $i$.

- Iteration 1: either $s \cdot t$ or the path $s \cdot 0 \cdot t$.
- It 2: all the paths between $s$ and $t$ that include 1 and perhaps 0, such as $s \cdot 1 \cdot t$, $s \cdot 1 \cdot 0 \cdot t$, and $s \cdot 0 \cdot 1 \cdot t$.
- Inductive hypothesis: The $i$th iteration of the loop sets the bit $(s, t)$ to true iff there is a directed path from $s$ to $t$ in the digraph that does not include any vertices with indices greater than $i$, (except possibly the endpoints $s$ and $t$).
• Assuming that it is true for the $i$th iteration of the loop, there is a path from $s$ to $t$ that does not include any vertices with indices greater than $i+1$ iff
  – (i) there is a path from $s$ to $t$ without indices $\geq i$, in which case $A[s][t]$ was set on a previous iteration of the loop (inductive hypothesis)
  – (ii) there is a path from $s$ to $i+1$ and a path from $i+1$ to $t$, neither of which includes any vertices with indices greater than $i$ (except endpoints), in which case $A[s][i+1]$ and $A[i+1][t]$ were previously set to true (by hypothesis), so the inner loop sets $A[s][t]$.

---

**Program 19.3. Warshall’s algorithm**

The constructor for class `GraphTC` computes the transitive closure of $A$ in the input data set $S$ so that the class can use `GraphTC` objects to see whether given vertices $a$ and $b$ are reachable from any other given vertex $c$. The constructor includes $T$ with a copy, $T = new GraphTC(A)$, which initializes itself with $T$ and $S$. We also use a `HashSet` object for the transitive closure $T$. Because the algorithm makes an efficient implementation of the edge entries, we use `HashSet`.

```java
class GraphTC {
    private HashSet<Edge> T;
    private GraphUtilities unionOrDifference(Graph G);
    T.insert(new Edge(a, b));
    for (int i = 0; i < S.size(); i++)
        T.insert(new Edge(a, i));
    return T;
}
```

• How to further improve?

• Test for $A[s][i]$ early
Finding the modules

Random Walk with 100000 steps

FINAL:

0.32589
0.30974
0.01614
0.23799
0.11024

Matrix multiplications with 10000 steps

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<th></th>
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<th>0.0163025758069772</th>
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</tr>
</tbody>
</table>

MCL clustering algorithm

- Markov (Chain Monte Carlo) Clustering
  - [http://www.micans.org/mcl/](http://www.micans.org/mcl/)
- Random walks according to edge weights
- Follow the different paths according to their probability
- Regions that are traversed “often” form clusters
What is Network Flow?

Flow network is a directed graph G=(V,E) such that each edge has a non-negative capacity c(u,v) ≥ 0.

Two distinguished vertices exist in G namely:
• Source (denoted by s) : In-degree of this vertex is 0.
• Sink (denoted by t) : Out-degree of this vertex is 0.

Flow in a network is an integer-valued function f defined on the edges of G satisfying 0 ≤ f(u,v) ≤ c(u,v), for every edge (u,v) in E.

Conditions for Network Flow

For each edge (u,v) in E, the flow f(u,v) is a real valued function that must satisfy following 3 conditions:
• Capacity Constraint: ∀ u,v ∈ V, f(u,v) ∈ [0,c(u,v)) (flow < capacity)
• Skew Symmetry: ∀ u,v ∈ V, f(u,v) = -f(v,u) (inflow = -outflow)
• Flow Conservation: ∀ u ∈ V - {s,t} Σ f(u,v) = 0 (net flow = 0)

Skew symmetry condition implies that f(u,u)=0.

The Value of a Flow

The value of a flow is given by:

\[ |f| = \sum_{s \in \text{source}} f(s,v) = \sum_{t \in \text{sink}} f(v,t) \]

The flow into the node is same as flow going out from the node and thus the flow is conserved. Also the total amount of flow from source s = total amount of flow into the sink t.
Example of a flow

Given a Graph G (V,E) such that:
- \( x_{ij} \) = flow on edge (i,j)
- \( u_{ij} \) = capacity of edge (i,j)
- \( s \) = source node
- \( t \) = sink node

Maximize \( v \)

Subject To
- \( \sum_j x_{ij} - \sum_j x_{ji} = 0 \) for each \( i \neq s, t \)
- \( \sum_j x_{sj} = v \)
- \( 0 \leq x_{ij} \leq u_{ij} \) for all \((i,j) \in E\).

In simple terms maximize the \( s \) to \( t \) flow, while ensuring that the flow is feasible.

Cuts of Flow Networks

A Cut in a network is a partition of \( V \) into \( S \) and \( T \) (\( T = V - S \)) such that
- \( s \) (source) is in \( S \) and \( t \) (target) is in \( T \).

Capacity of Cut (S,T)

The capacity of a cut \((S,T)\) is defined as the sum of the capacities of the edges going from \( S \) to \( T \).

Min Cut

Min \( s-t \) cut (Also called as a Min Cut) is a cut of minimum capacity.

Flow of Min Cut (Weak Duality)

Let \( f \) be the flow and let \((S,T)\) be a cut. Then \( |f| \leq \text{CAP}(S,T) \).

In maximum flow, minimum cut problems forward edges are full or saturated and the backward edges are empty because of the maximum flow. Thus maximum flow is equal to capacity of cut. This is referred to as weak duality.

Proof:

\[
|f| = \sum_{e \in S \rightarrow T} f(e) - \sum_{e \in T \rightarrow S} f(e) \\
= \sum_{e \in S \rightarrow T} u(e) - \sum_{e \in T \rightarrow S} u(e) \\
= \text{CAP}(S,T)
\]
Methods

Max-Flow Min-Cut Theorem

• The Ford-Fulkerson Method
• The Preflow-Push Method

The Ford-Fulkerson Method

Begin
x := 0; // x is the flow.
create the residual network G(x);
while there is some directed path from s to t in G(x) do
begin
let P be a path from s to t in G(x);
Δ := δ(P);
send Δ units of flow along P;
update the r’s;
end
end {the flow x is now maximum}.

Augmenting Paths (A Useful Concept)

Definition:
An augmenting path p is a simple path from s to t on a residual network
that is an alternating sequence of vertices and edges of the form
s, e₁, v₁, e₂, v₂, ..., eₖ, t in which no vertex is repeated and no forward edge
is saturated and no backward edge is free.

Characteristics of augmenting paths:
• We can put more flow from s to t through p.
• The edges of residual network are the edges on which residual capacity
is positive.
• We call the maximum capacity by which we can increase the flow on p
the residual capacity of p.

The Ford-Fulkerson’s Algorithm

FORDFULKERSON(G, E, s, t)
FOREACH e ∈ E
f(e) ← 0
Gₓ ← residual graph
WHILE (there exists augmenting path P)
f ← augment(f, P)
update Gₓ
ENDWHILE
RETURN f

Proof of correctness of the algorithm

Lemma: At each iteration all residual capacities are integers.
Proof: It’s true at the beginning. Assume it’s true after the first
it augmentation, and consider augmentation k along path P.
The residual capacity of P is the smallest residual capacity
on P, which is integral.
After updating, we modify the residual capacities by 0 or Δ,
and thus residual capacities stay integers.

Theorem: Ford-Fulkerson’s algorithm is finite
Proof: The capacity of each augmenting path is at least 1. The
augmentation reduces the residual capacity of some edge (u, j)
and doesn’t increase the residual capacity for some edge (u, i)
for any i.
So the sum of residual capacities of edges out of s keeps decreasing,
and is bounded below 0.
Number of augmentations is O(ΔC) where C is the largest of the
capacity in the network.
When is the flow optimal?

A flow $f$ is maximum flow in $G$ if:

1. The residual network $G_f$ contains no more augmented paths.
2. $|f| = c(S,T)$ for some cut $(S,T)$ (a min-cut)

Proof:

1. Suppose there is an augmenting path in $G_f$ then it implies that
   the flow $f$ is not maximum, because there is a path through which
   more data can flow. Thus if flow $f$ is maximum then residual n/w
   $G_f$ will have no more augmented paths.

2. Let $v=Fx(S,T)$ be the flow from s to t. By assumption $v=\text{CAP}(S,T)$
   By Weak duality, the maximum flow is at most $\text{CAP}(S,T)$. Thus the
   flow is maximum.

The Ford-Fulkerson Augmenting Path Algorithm for the Maximum Flow Problem

15.082 and 6.855J (MIT OCW)
Determine the capacity $\Delta$ of the path.
Send $\Delta$ units of flow in the path.
Update residual capacities.

Find any s-t path
Ford-Fulkerson Max Flow

Determine the capacity $\Delta$ of the path.
Send $\Delta$ units of flow in the path.
Update residual capacities.

There is no s-t path in the residual network. This flow is optimal

These are the nodes that are reachable from node s.

Here is the optimal flow

Counterexample for termination

Distribution & Transportation
Assigning teachers to classes

Teacher likes to teach C1, C4, C6
Every course will need a nr of teachers
Every teacher has a maximal capacity to teach
“Likes” – by weight

How would you solve it?

Job placement: 6 people, 6 jobs, preferences...

Converting the Matching problem to Network Flow

Converting Matching to Network Flow

Converting Optimal Bipartite Matching to Network Flow