Modelling and Control of Dynamic Systems

Stability of Linear Systems

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Motivating Example
The simplest way to control a linear system is to choose a compensator such that the output signal $y[k]$ tracks the reference signal $r[k - d]$:

$$\hat{y}[z] = \frac{1}{z^d} \cdot \hat{r}[z] \iff \hat{C}[z] \cdot \hat{G}[z] = \frac{1}{z^d} \cdot I .$$

However, such a controller is not perfect, as the physical process is likely to have disturbances and the system does not have to be initially in zero-state.
Design procedure

System identification

▷ Choose an appropriate model structure (fix a state space \( x \in \mathbb{R}^n \)).
▷ Determine analytically or experimentally all parameters of the system.

Controller design

▷ Compute the transfer function \( \hat{G}[z] \) for the parameters \( A, B, C, D \).
▷ Choose and implement the corresponding compensator \( \hat{C}[z] \).

Validation

▷ Run computer simulations to study the controllers behaviour.
▷ Run practical experiments to validate the design in practice.
An illustrative example

An open-loop controller for a simple feedback system

The behaviour of the system if $|\alpha| > 1$.

- The system can become uncontrollable.
- The controller cannot handle even mild disturbances.
- The controller cannot handle model estimation errors.
Stability of Discrete Systems
Stability of zero-state response

A linear system is *bounded-input bounded-output stable (BIBO stable)* if any bounded input signal \( u[\cdot] \) causes a bounded output signal \( y_{zs}[\cdot] \).

\( \triangleright \text{T1.} \) A SISO system is BIBO stable *iff* the impulse response sequence \( g[\cdot] \) is absolutely summable: \(|g[0]| + |g[1]| + |g[2]| + \cdots < \infty\).

\( \triangleright \text{T2.} \) Assume that a system with impulse response \( g[\cdot] \) is BIBO stable and consider the asymptotic behaviour in the process \( k \rightarrow \infty \).

\( \diamond \) Then the output \( y_{zs}[k] \) exited by \( u \equiv a \) approaches \( a \cdot \hat{g}[1] \).

\( \diamond \) Then the output \( y_{zs}[k] \) exited by a sinus signal \( u[k] = \sin(\omega_0 k) \) approaches to a sinus signal with the same frequency:

\[
y_{zs}[k] \approx \hat{g}[e^{i\omega_0}] \sin(\omega_0 k + \angle g[e^{i\omega_0}]) .
\]
BIBO stability and transfer function

\( \triangleright \textbf{T3.} \) A \textit{continuous} linear system is BIBO stable \textit{iff} every pole \( \hat{g}(s) \) lies in the left-half plane \( (\Re(s) < 0) \). A \textit{discrete} linear system is BIBO stable \textit{iff} every pole \( \hat{g}[z] \) has lies inside the unit circle \( (|z| < 1) \).
A MIMO system is BIBO stable if every sub-component is BIBO stable.
Stability of zero-input response

The zero-input response of the equation $x[k + 1] = Ax[k]$ is **marginally stable** if every initial state $x_0$ excites a bounded response $x[\cdot]$. The zero-input response is **asymptotically stable** if every initial state $x_0$ excites a bounded response $x[\cdot]$ that approaches 0 as $k \to \infty$.

▷ **C1.** The zero-input response $y_{zi}[\cdot]$ of a marginally stable system is bounded. If the system is asymptotically stable then $y_{zi}[k] \to 0$.

▷ **C2.** A BIBO stable open-loop controller does not cause catastrophic consequences if the system is BIBO and asymptotically stable.

▷ **R1.** Not all realisations of BIBO stable systems are marginally or asymptotically stable, since some state variables might be *unobservable*. 
Stability of state equations

A *minimal polynomial* of a matrix $A$ is a polynomial $f(\lambda)$ with minimal degree such that $f(A) = f_0 \cdot A^k + f_1 \cdot A^{k-1} + \cdots + f_k \cdot A^0 = 0$.

- **T4.C** The equation $\dot{x}(t) = Ax(t)$ is *asymptotically stable* iff all eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ satisfy $\Re(\lambda_i) < 0$. The equation $\dot{x} = Ax$ is *marginally stable* iff all eigenvalues satisfy $\Re(\lambda_i) \leq 0$ and all eigenvalues with $\Re(\lambda_i) = 0$ are simple roots of the minimal polynomial of $A$.

- **T4.D** The equation $x[k+1] = Ax[k]$ is *asymptotically stable* iff all eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ satisfy $|\lambda_i| < 1$. The equation $x[k+1] = Ax[k]$ is *marginally stable* iff all eigenvalues satisfy $|\lambda_i| \leq 1$ and all eigenvalues with $|\lambda_i| = 1$ are simple roots of the minimal polynomial of $A$. 
Stability of minimal realisations

A realisation of a transfer function $\hat{G}[z]$ is minimal if the state equation

$$
\begin{align*}
x[k+1] &= Ax[k] + Bu[k] \\
y[k] &= Cx[k] + Du[k]
\end{align*}
$$

has a state space with minimal dimension.

- **T5.** Consider a minimal realisation of a transfer function $\hat{g}[z]$. Then all eigenvalues of $A$ are poles of $\hat{g}[z]$ and vice versa.

- **R2.** Poles of a transfer function $\hat{g}[z]$ are always eigenvalues of $A$.

- **C4.** A minimal realisation of a transfer function $\hat{g}[z]$ is *asymptotically stable* iff the transfer function is BIBO stable.