Modelling and Control of Dynamic Systems

Basic Concepts

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What is a dynamical system?

<table>
<thead>
<tr>
<th>Continuous-time system</th>
<th>Discrete-time system</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(t)$ → $x(t)$ → $y(t)$</td>
<td>$u[t]$ → $x[t]$ → $y[t]$</td>
</tr>
<tr>
<td>Signals are sampled continuously</td>
<td>Signals are sampled at finite rate</td>
</tr>
</tbody>
</table>

**Notation:**
- $u(t)$ and $u[t]$ denote input signals
- $y(t)$ and $y[t]$ denote output signals
- $x(t)$ and $x[t]$ denotes the internal state of a system
Examples of dynamic systems

- Various physical objects: cars, planes, electronic circuits, ...
- Chip-on-chip measurements

![Intensity vs. Location in the DNA strand](image)
A dynamic system is *memoryless* if its output $y(t)$ depends only on the input $u(t)$ for all sampled time points $t \in \mathbb{R}$.

If a systems has a memory then its output $y(t_0)$ can depend on
- on the current input $u(t)$;
- on past inputs $u(t)$ and outputs $y(t)$ where $t < t_0$;
- on future inputs $u(t)$ and outputs $y(t)$ where $t > t_0$.

A dynamic system is *causal* if its output $y(t_0)$ does not depend on the future inputs $u(t)$ and outputs $y(t)$.

For any deterministic causal system, the current output $y(t_0)$ is *uniquely* determined by the infinite range of values $[u(t) : t \leq t_0]$ and $[v(t) : t < t_0]$. 
What is a state?

The state $x(t_0)$ of a system at time $t_0$ is the information that together with the input $u(t)$ for $t > t_0$ determines uniquely the output $y(t)$ for all $t \geq t_0$.

♣ For randomised systems, it is reasonable to exclude the randomness $\omega \in \Omega$ from the state. Hence, the state $x(t_0)$ determines only the probability distribution over all possible output traces $[y(t) : t \geq t_0]$.

A dynamic system is \textit{lumped} if its number of state variables is finite.

▷ A modern computer is lumped system.

▷ Dynamics of a car can be modelled with a lumped system.

A dynamic system is \textit{distributed} if its number of state variables is infinite.

▷ A simple delay unit $y(t) = u(t - 1)$ is distributed system.

▷ Distributed systems are scarce in nature if non-existent.
Illustrating examples

A BALL ON A PLANE

▷ What is the output vector?
▷ What are the input signals?
▷ What is the state of the system?

A CELL IN A TEST TUBE

▷ What is the output vector?
▷ What are the input signals?
▷ What is the state of the system?
Time-invariant and time-varying systems

A dynamic causal system with a state $x(t)$ is time invariant if for all possible input states, input signals $u(t)$ and time shifts $\Delta t$ conditions

\[
\begin{align*}
\begin{cases}
  x(t_0), \\
  u(t), \quad t > t_0
\end{cases}
\quad \text{and} \quad \\
\begin{cases}
  x(t_0 + \Delta t), \\
  u(t - \Delta t), \quad t > t_0 + \Delta
\end{cases}
\end{align*}
\]

determine output signals $y_1(t)$ and $y_2(t)$ that are shifted:

\[
y_1(t) = y_2(t + \Delta t), \quad t > t_0.
\]

Informally, the dynamics of a system does not change in time.