Problem 1: Deutsch-Jozsa Algorithm

Assume that $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is a function that satisfies one of the following two properties:

- $f$ is constant (i.e., $f(x) = f(y)$ for all $x, y \in \{0, 1\}^n$), or
- $f$ is balanced (i.e., $|\{x : f(x) = 0\}| = |\{x : f(x) = 1\}| = 2^{n-1}$).

That is, we have the promise that $f$ is constant or balanced, but we do not know which of the two holds.

Let $U_f$ be the unitary transformation on $\mathbb{C}^{2^n}$ defined by

$$U_f| x, y \rangle = | x, y \oplus f(x) \rangle \quad (x \in \{0, 1\}^n, y \in \{0, 1\}).$$

Consider the following circuit:

|0...0⟩ $\xrightarrow{H \otimes n}$ $\xrightarrow{U_f}$ $\xrightarrow{H \otimes n}$ $\xrightarrow{M}$ |Ψ1⟩ $\xrightarrow{H}$ |Ψ2⟩ $\xrightarrow{H \otimes n}$ |Ψ3⟩ $\xrightarrow{H \otimes n}$ |Ψ4⟩

where $M$ is a complete measurement in the computational basis.

The $|\Psi_i\rangle$ denote the intermediate states after the individual steps of the algorithm. E.g., $|\Psi_1\rangle = |0...01\rangle$.

(a) What is $|\Psi_2\rangle$?

(b) Show that

$$|\Psi_3\rangle = \sum_{x \in \{0, 1\}^n} 2^{-n/2 - 1/2} |x, f(x)\rangle - 2^{-n/2 - 1/2} |x, \overline{f(x)}\rangle.$$  
(Here $\overline{f(x)} := 1 - f(x)$.)

(c) Show that

$$|\Psi_3\rangle = \left(2^{-n/2} \sum_{x \in \{0, 1\}^n} (-1)^{f(x)} |x\rangle \right) \otimes |\rangle$$

Here $|\rangle$ is short for $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$. 
(d) Show that \( H^\otimes n |x\rangle = 2^{-n/2} \sum_{z \in \{0,1\}^n} (-1)^{x \cdot z} |z\rangle \) where \( x \cdot z := \sum_{i=1}^n x_i z_i \).

(e) What is \(|\Psi_4\rangle\)?

(f) Show that the probability \( P \) of measuring 0\ldots0 in the measurement is 
\[ \left(2^{-n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \right)^2. \]

(g) Compute the probability \( P \) of measuring 0\ldots0 in the case that \( f \) is constant.

(h) Compute the probability \( P \) of measuring 0\ldots0 in the case that \( f \) is balanced.

**Problem 2: Ensembles and Density Operators**

(a) Consider the following quantum ensembles:
\[
E_1 = \{(|0\rangle, \frac{1}{2}), (|+\rangle, \frac{1}{2})\}, \\
E_2 = \{(|0\rangle, \frac{1}{4}), (|1\rangle, \frac{3}{4})\}, \\
E_3 = \{(|0\rangle, \frac{1}{4}), (|1\rangle, \frac{1}{4}), (|+\rangle, \frac{1}{4}), (|\rangle, \frac{1}{4})\}.
\]
Compute the corresponding density operators \( \rho_1, \rho_2, \rho_3 \) as explicitly given matrices. (Note: \(|+\rangle := \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \) and \(|\rangle := \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \).)

(b) Consider the following process: First, a random value \( x \in \{0,1\}^n \) is chosen. Then an \( n \)-bit quantum register is prepared to have the value \( |\Psi\rangle := |x\rangle \). Then a unitary transformation \( U \) is applied to \( \Psi \). What is the density operator corresponding to the resulting ensemble?

- **Hint:** As the first step, consider the case that \( U \) is the identity.

(c) Let a measurement \( M \) consisting of projectors \( P_1, \ldots, P_n \) be given. Let a quantum state \(|\Psi\rangle\) be given. Assume that \(|\Psi\rangle\) is measured using \( M \) but the measurement outcome is not recorded (i.e., it is forgotten, erased). What is the ensemble describing the state of the system after this experiment? What is the corresponding density operator?

- **Note:** The formula in the lecture was for the case where the measurement outcome is forgotten.

(d) Assume a quantum system is in the state described by a density operator \( \rho \). We apply a measurement \( M \) consisting of projectors \( P_1, \ldots, P_n \) to the system and forget the outcome. What is the density operator describing the resulting state of the system?

(e) In the lecture, we mentioned several times that a global phase, i.e., a factor \( \varphi \in \mathbb{C} \) with \(|\varphi| = 1\) in front of a quantum state, is physically irrelevant. Demonstrate this by showing that the two states \(|\Psi\rangle\) and \(\varphi|\Psi\rangle\) are physically indistinguishable. \[ ^1 \text{More precisely, that the ensembles \{(|\Psi\rangle, 1)\} and \{(\varphi|\Psi\rangle, 1)\} are physically indistinguishable.} \]
Problem 3: Physical indistinguishability – the opposite direction (bonus problem)

Let $E_1$ and $E_2$ be ensembles with density matrices $\rho_1$ and $\rho_2$. Assume that $\rho_1 \neq \rho_2$. Prove that $E_1$ and $E_2$ are physically distinguishable by specifying a measurement $M = \{Q_{\text{yes}}, Q_{\text{no}}\}$ with the following property: When measuring $E_1$ and $E_2$ with $M$, we get the outcome yes with different probabilities $P_1$ and $P_2$ (where $P_i := \Pr[\text{Outcome is yes when measuring } \rho_i]$).

Hint: Consider the matrix $\sigma := \rho_1 - \rho_2$. Show that $\sigma$ is diagonalisable and that it therefore has an eigenvector $|\Psi\rangle$ with eigenvalue $\lambda \neq 0$. Set $Q_{\text{yes}} := |\Psi\rangle\langle\Psi|$. You may use without proof the fact that a density operator is always Hermitian.