Quantum Cryptography (fall 2017) Dominique Unruh

Exercise Sheet 4

Out: 2017-10-16 Due: 2017-10-23

Problem 1: Deutsch-Jozsa Algorithm

Assume that $f : \{0,1\}^n \to \{0,1\}$ is a function that satisfies one of the following two properties:

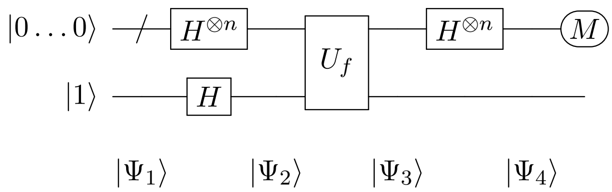
- f is constant (i.e., $f(x) = f(y)$ for all $x, y \in \{0,1\}^n$), or
- f is balanced (i.e., $|\{x : f(x) = 0\}| = |\{x : f(x) = 1\}| = 2^{n-1}$).

That is, we have the promise that f is constant or balanced, but we do not know which of the two holds.

Let U_f be the unitary transformation on $\mathbb{C}^{2^n}$ defined by

$$U_f|x, y\rangle = |x, y \oplus f(x)\rangle \qquad (x \in \{0,1\}^n, y \in \{0,1\}).$$

Consider the following circuit:



where M is a complete measurement in the computational basis.

The $|\Psi_i\rangle$ denote the intermediate states after the individual steps of the algorithm. E.g., $|\Psi_1\rangle = |0 \dots 01\rangle$.

(a) What is $|\Psi_2\rangle$?

(b) Show that

$$|\Psi_3\rangle = \sum_{x \in \{0,1\}^n} 2^{-n/2-1/2}|x, f(x)\rangle - 2^{-n/2-1/2}|x, \overline{f(x)}\rangle.$$

(Here $\overline{f(x)} := 1 - f(x)$.)

(c) Show that

$$|\Psi_3\rangle = \Big(2^{-n/2} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}|x\rangle\Big) \otimes |-\rangle$$

Here $|-\rangle$ is short for $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.

(d) Show that $H^{\otimes n}|x\rangle = 2^{-n/2}\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle$ where $x\cdot z := \sum_{i=1}^n x_i z_i$.

(e) What is $|\Psi_4\rangle$?

(f) Show that the probability P of measuring $0\dots0$ in the measurement is $\bigl(2^{-n}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\bigr)^2$.

(g) Compute the probability P of measuring $0\dots0$ in the case that f is constant.

(h) Compute the probability P of measuring $0\dots0$ in the case that f is balanced.

Problem 2: Ensembles and Density Operators

(a) Consider the following quantum ensembles:

$$\begin{aligned}
E_1 &= \{(|0\rangle, \tfrac{1}{2}), (|+\rangle, \tfrac{1}{2})\},\\
E_2 &= \{(|0\rangle, \tfrac{1}{4}), (|1\rangle, \tfrac{3}{4})\},\\
E_3 &= \{(|0\rangle, \tfrac{1}{4}), (|1\rangle, \tfrac{1}{4}), (|+\rangle, \tfrac{1}{4}), (|-\rangle, \tfrac{1}{4})\}.
\end{aligned}$$

Compute the corresponding density operators ρ_1, ρ_2, ρ_3 as explicitly given matrices. (Note: $|+\rangle := \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle := \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.)

(b) Consider the following process: First, a random value $x \in \{0,1\}^n$ is chosen. Then an n-bit quantum register is prepared to have the value $|\Psi\rangle := |x\rangle$. Then a unitary transformation U is applied to Ψ. What is the density operator corresponding to the resulting ensemble?

Hint: As the first step, consider the case that U is the identity.

(c) Let a measurement M consisting of projectors $P_1, \dots, P_n$ be given. Let a quantum state $|\Psi\rangle$ be given. Assume that $|\Psi\rangle$ is measured using M but the measurement outcome **is not recorded** (i.e., it is forgotten, erased). What is the ensemble describing the state of the system after this experiment? What is the corresponding density operator?

Note: The formula in the lecture was for the case where the measurement outcome is **not** forgotten.

(d) Assume a quantum system is in the state described by a density operator ρ. We apply a measurement M consisting of projectors $P_1, \dots, P_n$ to the system and forget the outcome. What is the density operator describing the resulting state of the system?

(e) In the lecture, we mentioned several times that a global phase, i.e., a factor $\varphi \in \mathbb{C}$ with $|\varphi| = 1$ in front of a quantum state, is physically irrelevant.

Demonstrate this by showing that the two states $|\Psi\rangle$ and $\varphi|\Psi\rangle$ are physically indistinguishable.[1]

[1] More precisely, that the ensembles $\{(|\Psi\rangle, 1)\}$ and $\{(\varphi|\Psi\rangle, 1)\}$ are physically indistinguishable.

Problem 3: Physical indistinguishability – the opposite direction (bonus problem)

Let E_1 and E_2 be ensembles with density matrices ρ_1 and ρ_2. Assume that $\rho_1 \neq \rho_2$. Prove that E_1 and E_2 are physically distinguishable by specifying a measurement $M = \{Q_{\text{yes}}, Q_{\text{no}}\}$ with the following property: When measuring E_1 and E_2 with M, we get the outcome yes with different probabilities P_1 and P_2 (where $P_i :=$ Pr[Outcome is yes when measuring ρ_i]).

Hint: Consider the matrix $\sigma := \rho_1 - \rho_2$. Show that σ is diagonalisable and that it therefore has an eigenvector $|\Psi\rangle$ with eigenvalue $\lambda \neq 0$. Set $Q_{\text{yes}} := |\Psi\rangle\langle\Psi|$. You may use without proof the fact that a density operator is always Hermitean.