Quantum Cryptography (fall 2017)	Dominique Unruh
Exercise Sheet 1	
Out: 2017-09-11	Due: 2017-09-25

You will need 50% of all homeworks to qualify for the exam. (That is, if you get at least 50%, your final grade will be the exam grade. And if you do not get 50%, you do not pass the course.)

You may hand in your solutions in person or by email. If you submit by email, either scan a handwritten solution or typeset your solution readably. I do not consider ASCII formulas readable. For nicely typeset solutions, you can get up to three extra points for the effort.

When submitting, indicate your name and your matriculation number. On your first submission, please also indicate a password, this password will be needed for accessing the solutions and your points online.

You may work in teams to solve the problems. If you do, everyone has to formulate their own solution! (No copy&paste.)

Problem 1: Qubits

(a) Which of the following are valid quantum states:

$$|1\rangle, \quad |0\rangle + |1\rangle, \quad \tfrac{1}{\sqrt{2}}(|0\rangle - i|1\rangle), \quad \tfrac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |1\rangle), \quad \sqrt{\tfrac{2}{3}}|0\rangle + \tfrac{i}{\sqrt{3}}|1\rangle.$$

(b) For each of the *valid* quantum states from (a), answer the following: You perform a measurement in basis $|0\rangle, |1\rangle$ (i.e., you ask "whether the state is $|0\rangle$ or $|1\rangle$). What is the probability of answer 0 (i.e., yes), what is the probability of answer 1 (i.e., no)? What is the state after the measurement in each of those cases?

(c) For each of the *valid* quantum states from (a), answer the following: You perform a measurement in basis $|+\rangle, |-\rangle$. What is the probability of answer + (i.e., yes), what is the probability of answer − (i.e., no)? What is the state after the measurement in each of those cases?

(d) Let a quantum state $|\Psi\rangle \in \mathbb{C}^2$ and an (orthonormal) measurement basis $|\mathrm{yes}\rangle, |\mathrm{no}\rangle \in \mathbb{C}^2$ be given. Measure $|\Psi\rangle$ in that measurement basis. Let P_{yes} be the probability of outcome yes, and P_{no} the probability of outcome no. Show that $P_{\mathrm{yes}} + P_{\mathrm{no}} = 1$.

(e) Show that by applying a unitary transformation to a quantum state, no information is ever lost. More exactly, assume that a unitary transformation U is applied to a given quantum state $|\Psi\rangle$, resulting in a state $|\Phi\rangle$. Then show that there is another unitary transformation V (not depending on $|\Psi\rangle$ or $|\Phi\rangle$) such that applying V to $|\Phi\rangle$ gives $|\Psi\rangle$ again.

(f) Assume that a photon is in the state $|\Psi\rangle = \alpha|\updownarrow\rangle + \beta|\leftrightarrow\rangle$. Let R be a rotation of angle $\theta = \frac{\pi}{3}$. Let F denote a polarisation filter that lets only vertically polarised light through ($|\updownarrow\rangle$). Assume that the photon $|\Psi\rangle$ is first sent through R and then through F. It turns out that in this setting, the photon is absorbed by F with probability 1.

Given these informations, what do you know about α? (I.e., what are the possible values of α?)

(g) What is wrong with the following approach:

Alice has a qubit $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. She wants to initialise the qubit to $|0\rangle$. She knows that when measuring $|\Psi\rangle$ in the computational basis $|0\rangle, |1\rangle$, with probability $\frac{1}{2}$ she get the measurement outcome 0 and the qubit will be in state $|0\rangle$. Thus she repeatedly measures the qubit in the computational basis until she gets the outcome 0. Since the probability is $\frac{1}{2}$ each time, the expected number of measurements until she gets her $|0\rangle$-initialised qubit is 2.

(h) Which of the following are valid (unitary) transformations:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}i & -1\\ i & 1\end{pmatrix}.$$