Problem 1: Schmidt Decomposition

(a) For a given state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, the Schmidt number is the smallest $n$ such that a Schmidt decomposition $|\Psi\rangle = \sum_{i=1}^{n} \lambda_i |\alpha_i\rangle |\beta_i\rangle$ exists.

We call a state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ entangled if $|\Psi\rangle$ cannot be written as $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$.

Show that a state is entangled if and only if it has Schmidt number greater than 1. (This justifies using the Schmidt number as a measure of how entangled a state is.)

(b) Let a state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be given. Assume for simplicity that $\dim \mathcal{H}_A = \dim \mathcal{H}_B$. Show that $\text{tr}_A |\Psi\rangle \langle \Psi|$ and $\text{tr}_B |\Psi\rangle \langle \Psi|$ have the same eigenvalues.

Hint: Represent $|\Psi\rangle$ in its Schmidt decomposition. Then compute the partial trace $\text{tr}_A$ and $\text{tr}_B$ directly on that representation.