Problem 1: Quantum Operations

Describe the partial trace as a quantum operation. More exactly, let $\mathcal{H}_A = \mathbb{C}^n$, $\mathcal{H}_B = \mathbb{C}^m$. Find operators $E_k : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A$ such that these define a quantum operation $\mathcal{E} = \{E_k\}_k$ with the property that $\mathcal{E}(\rho) = \text{tr}_B \rho$ for all $\rho$. Show that $\mathcal{E}$ is indeed a quantum operation (i.e., that the $E_k$ are valid operators for defining a quantum operation).

Hint: For density operators $\rho$ we have $\text{tr} \rho = \sum_k \langle k | \rho | k \rangle$. Note that here $\langle k |$ is a linear operator from $\mathcal{H}_B$ to $\mathbb{C}$. And $I \otimes \langle k |$ is a linear operator from $\mathcal{H}_A \otimes \mathcal{H}_B$ to $\mathcal{H}_A \otimes \mathbb{C} = \mathcal{H}_A$.

Problem 2: Purification

Compute purifications of the following density operators:

$\rho_1 := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2 := \frac{1}{2} |00\rangle \langle 00| + \frac{1}{2} |11\rangle \langle 11|, \quad \rho_3 := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Hint: Eigenvectors of $\rho_3$ are $|+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$.

Problem 3: Trace Distance

(a) Let $E_1$ and $E_2$ be quantum ensembles. Let $\rho_1$ and $\rho_2$ be the corresponding density operators. Assume that $E_1$ and $E_2$ are physically indistinguishable. What is $\text{TD}(\rho_1, \rho_2)$?

(b) Let $E_1 := \{(|+\rangle, \frac{1}{2}), (|\rangle, \frac{1}{2})\}$ and $E_2 := \{|0\rangle, 1\}$ be quantum ensembles. Let $\rho_1$ and $\rho_2$ be the corresponding density operators. What is $\text{TD}(\rho_1, \rho_2)$?

(c) Let $\rho = p\tau + q\rho'$ and $\sigma = p\tau + q\sigma'$ where $\tau, \rho', \sigma'$ are density operators, and $p, q \geq 0$, $p + q = 1$. Show that $\text{TD}(\sigma, \rho) = q \cdot \text{TD}(\sigma', \rho')$.

Note: Do not use Lemma 9 in the lecture notes.

(d) Let $E_1 := \{(|+\rangle, \frac{1}{2}), (|\rangle, \frac{1}{2}), (|\Psi\rangle, \frac{1}{2})\}$. Let $E_2 := \{|0\rangle, \frac{1}{2}, (|\Psi\rangle, \frac{1}{2})\}$. Here $|\Psi\rangle := \frac{1}{\sqrt{2}} |0\rangle - \sqrt{\frac{3}{4}} |1\rangle$. Let $\rho_1$ and $\rho_2$ be the corresponding density operators. What is $\text{TD}(\rho_1, \rho_2)$?

Hint: Consider (c).
(e) Consider the following setup: Alice has a secret bit $b \in \{0, 1\}$. Then she chooses randomly $r \in \{0, 1\}$. If $r = 0$, she encodes $b$ in the $|0\rangle, |1\rangle$ basis (i.e., she sends $|0\rangle$ for $b = 0$ and $|1\rangle$ for $b = 1$). If $r = 1$, she encodes $b$ in the $|+\rangle, |−\rangle$ basis. Then she sends the resulting state $|\Psi_b\rangle$ to Eve. Show that the trace distance between the mixed states $\rho_0$ and $\rho_1$ corresponding to $b = 0$ and $b = 1$, respectively, is $\text{TD}(\rho_0, \rho_1) = \frac{1}{\sqrt{2}}$.

**Hint:** The eigenvalues of $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ are $\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$. Note that this is not the toy protocol from the lecture, in the toy protocol $b$ selected the basis, not $r$.

(f) In the experiment described in (e), assume that the bit $b$ is chosen uniformly at random. Show that Eve cannot guess $b$ with probability larger than $\frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 85\%$.

**Hint:** Try to express the probability that Eve guesses correctly in terms of $\text{Pr}[G = x | b = y]$ for various $x, y \in \{0, 1\}$ (here $G$ denotes Eve’s guess) and then use (e).