Problem 1: Deterministic PCPs

PCP stands for “probabilistically checkable proofs”. We will now show that the “probabilistic” part is very important for the definition. Namely, we show that $\text{PCP}(0, \log) = \text{P}$ and $\text{PCP}(0, \text{poly}) = \text{NP}$.

(a) Show that $\text{P} \subseteq \text{PCP}(0, \log)$. (In fact, it is even in $\text{PCP}(0, 0)$.)

(b) Show that $\text{NP} \subseteq \text{PCP}(0, \text{poly})$.

(c) Consider a sound and complete $(0, q)$-PCP-verifier $V$ for some language $L$ (i.e., a deterministic verifier). Construct a polynomial-time deterministic algorithm $V'(x, w)$ such that:

- If $x \in L$ then $\exists w \in \{0, 1\}^{q(|x|)}. V'(x, w) = 1$
- If $x \notin L$ then $\forall w \in \{0, 1\}^{q(|x|)}. V'(x, w) = 0$

*Hint:* $V$ already kind-of satisfies this, except that $V$ gets a longer input than $q$ bits. But $V$ only looks at $q$ bits of that input.

(d) Show that $\text{PCP}(0, \text{poly}) \subseteq \text{NP}$.

*Hint:* This follows quite easily from (c).

(e) Show that $\text{PCP}(0, \log) \subseteq \text{P}$.

*Hint:* Given the verifier $V'$ from (c), how can you can whether $V'(x, w) = 1$ for all $w$? How long does that take?

Problem 2: Approximating MAX-SAT is NP-hard

In the following, let $q \geq 2$ be an integer.

(a) Let $\varphi$ be a $q$CSP. That is, $\varphi = \varphi_1 \land \cdots \land \varphi_t$, where each $\varphi_i$ is a formula containing at most $q$ different variables.
(b) Show that $\varphi$ can be transformed into a $q$CNF formula with at most $2^q t$ clauses (in polynomial time in the size of $\varphi$) such that $\varphi$ is satisfiable iff $\varphi'$ is satisfiable.

**Note:** If you are troubled whether a runtime factor of $2^q$ is OK for a polynomial-time algorithm, recall that $q$ is a constant in this context, so $2^q$ is, as well.

(c) Show: If $\text{val}(\varphi) \leq 1 - \varepsilon$ (for any assignment at least an $\varepsilon$-fraction of the clauses is violated), then $\text{val}(\varphi') \leq 1 - \varepsilon/2^q$ (for any assignment at least an $\varepsilon/2^q$-fraction of the clauses is violated).

(d) Assume there is a polynomial-time algorithm $A$ that approximates MAX-SAT up to a factor of $\rho := 1 - 2^{-q-2}$.

(That is, given a formula $\varphi'$ in CNF it returns a value $\delta$ such that $\rho \cdot \text{val}(\varphi') \leq \delta \leq \text{val}(\varphi')$.)

Show that there is a polynomial-time algorithm $B$ such that for any $q$CSP $\varphi$, we have: If $\text{val}(\varphi) = 1$, then $B(\varphi) = 1$. If $\text{val}(\varphi) \leq \frac{1}{2}$, then $B(\varphi) = 0$.

**Note:** If you manage to prove this with another approximation factor $\rho$, that is also fine, as long as $0 < \rho < 1$.

(e) Assume that $P \neq NP$. Show that an constant $0 < \rho < 1$ such that no polynomial-time algorithm $A$ approximates MAX-SAT up to a factor of $\rho := 1 - 2^{-q-2}$.

**Hint:** Use a theorem from the lecture here, together with [4].