Problem 1: $\Sigma^p_I$-completeness

Let

$$\Sigma^p_{\text{formulaSAT}} := \{ \varphi : \exists u_1 \forall u_2 \exists u_3 \ldots \varphi(u_1, \ldots, u_i) = 1 \}$$

where $\varphi$ is a Boolean circuit with $i$ multi-bit inputs. (In particular, $|u_i| \leq |\varphi|$ for all $i$.)

Show: $\Sigma^p_{\text{formulaSAT}}$ is $\Sigma^p_I$-complete.

Note: You can take the following facts for granted (i.e., you do not need to reprove them): Given a Turing machine $M(x, u_1, \ldots, u_i)$ and given integers $\ell_x, \ell_1, \ldots, \ell_i$ indicating the lengths of the different inputs, one can construct a circuit $\varphi$ in polynomial-time (in $\ell_x + \sum \ell_i$) such that $M(x, u_1, \ldots, u_i) = \varphi(x, u_1, \ldots, u_i)$ for all $x, u_1, \ldots, u_i$ of the specified lengths. Furthermore, given $\varphi$ and inputs for $\varphi$, one can compute the output of $\varphi$ with a polynomial-time TM.

Hint: The problem is easier when you use the alternative definition of $\Sigma^p_I$ (Definition 5.3 in Arora-Barak).

We restate the definition here:

**Definition 1** $L \in \Sigma^p_I$ iff there exists a polynomial-time TM $M$ and a polynomial $q$ such that for all $x$:

$$x \in L \iff \exists u_1 \in \{0, 1\}^{q(|x|)} \forall u_2 \in \{0, 1\}^{q(|x|)} \exists u_3 \in \{0, 1\}^{q(|x|)} \ldots M(x, u_1, \ldots, u_i) = 1$$

Problem 2: The collapse of the polynomial-hierarchy

(a) Show: If any two of $\Sigma^p_1, \Sigma^p_2, \ldots, \Pi^p_1, \Pi^p_2, \ldots$ are equal, then the polynomial-hierarchy collapses.

Note: In the practice we already showed that if $\Sigma^p_1 = \Pi^p_1$, the polynomial-hierarchy collapses. (You can use that fact.) But you need to show that the same holds for any other two classes, too. (Also $\Sigma$’s with $\Sigma$’s, different levels, etc.)

(b) Show: If graph isomorphism is NP-complete, then there is a PH-complete language.

Note: You may use all facts that were mentioned in the lecture, even those mentioned without proof. Also facts from other problems in this homework. No complex proofs are required for this problem, it follows easily from the right facts.