Problem 1: Malleability of ElGamal

Remember the auction example from the lecture: Bidder 1 produces a ciphertext $c = E(pk, bid_1)$ where $E$ is the ElGamal encryption algorithm (using integers mod $p$ as the underlying group). Given $c$, Bidder 2 can then compute $c'$ such that $c'$ decrypts to $2 \cdot bid_1 \mod p$. This allows Bidder 2 to consistently bid twice as much as Bidder 1.

Now refine the attack. You may assume that $bid_1$ is the amount of Cents Bidder 1 is willing to pay. And you can assume that Bidder 1 will always bid a whole number of Euros. (I.e., $bid_1$ is a multiple of 100.)

Show how Bidder 2 can consistently overbid Bidder 1 by only 1%. What happens to your attack if Bidder 1 suddenly does not bid a whole number of Euros?

**Hint:** Remember that modulo $p$, one can efficiently find inverses. For example, one can find a number $a$ such that $a \cdot 100 \equiv 1 \mod p$.

Problem 2: Encoding messages for ElGamal (bonus problem)

The message space of ElGamal (when using the instantiation that operates modulo a prime $p > 2$ with $p \equiv 3 \mod 4$ and if we want to avoid the insecurity discussed in the practice) is the set $\mathbb{QR}_p = \{x^2 \mod p : x = 0, \ldots, p-1\}$.

The problem is now: if we wish to encrypt a message $m \in \{0, 1\}^\ell$ (with $\ell \leq |p| - 2$), how do we interpret $m$ as an element of $\mathbb{QR}_p$?

One possibility is to use the following function $f : \{1, \ldots, \frac{p-1}{2}\} \to \mathbb{QR}_p$:

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{QR}_p \\ -x \mod p & \text{if } x \notin \mathbb{QR}_p \end{cases}$$

Once we see that $f$ is a bijection and can be efficiently inverted, the problem is solved, because a bitstring $m \in \{0, 1\}^\ell$ can be interpreted as a number in the range $1, \ldots, \frac{p-1}{2}$ by simply interpreting $m$ as a binary integer and adding 1 to it. (I.e., we encrypt $f(m + 1)$.)

We claim that the following function is the inverse of $f$:

$$g(x) := \begin{cases} x & \text{if } x = 1, \ldots, \frac{p-1}{2} \\ -x \mod p & \text{if } x \notin 1, \ldots, \frac{p-1}{2} \end{cases}$$

We thus need to show the following: the range of $f$ is indeed $\mathbb{QR}_p$, and that $g(f(x)) = x$ for all $x \in \{1, \ldots, \frac{p-1}{2}\}$.

(a) Show that $f(x) \in \mathbb{QR}_p$ for all $x \in \{1, \ldots, \frac{p-1}{2}\}$.

**Hint:** You can use (without proof) that $-1 \notin \mathbb{QR}_p$ (this only holds in $\mathbb{QR}_p$ for $p$ prime with $p \equiv 3 \mod 4$). And that the product of two quadratic non-residues is a quadratic residue (this only holds in $\mathbb{QR}_p$, but not in $\mathbb{QR}_n$ for $n$ non-prime).

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1. As long as $bid_1 < p/2$, that is. Otherwise $2 \cdot bid_1 \mod p$ will not be twice as much as $bid_1$. However, for large $p$, $bid_1 \geq p/2$ is an unrealistically high bid.
2. You do not actually need to use this fact, but the hint that $-1 \notin \mathbb{QR}_p$ below is only true in this case.
(b) Show that \( g(f(x)) = x \) for all \( x \in \{1, \ldots, \frac{p-1}{2}\} \).

(This then shows that \( f \) is injective and efficiently invertible. Bijectivity follows from injectivity because the domain and range of \( f \) both have the same size.)

\textbf{Hint:} Make a case distinction between \( x \in \text{QR}_p \) and \( x \not\in \text{QR}_p \). Show that for \( x \in \{1, \ldots, \frac{p-1}{2}\} \) it holds that \(-x \mod p \not\in \{1, \ldots, \frac{p-1}{2}\} \).