MTAT.05.128: Introduction to Digital Communications **University of Tartu**

Homework assignment 5

Due date: May 25, 2016

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1. Consider a Gaussian random variable $X \sim \mathcal{N}(0,1)$. Define a new random variable $Z = b \cdot (X + a)$, where $a, b \in \mathbb{R}$. Show that for some $\alpha, \beta \in \mathbb{R}$: $Z \sim \mathcal{N}(\alpha, \beta)$. Find these α and β.

2. The input to a noisy communications channel is a random variable X taking values in $\{-3, -1, 1, 3\}$, defined by the following probability distribution:

$$\mathsf{Pr}(X = -3) = \mathsf{Pr}(X = -1) = \mathsf{Pr}(X = 1) = \mathsf{Pr}(X = 3) = \frac{1}{4} \,.$$

   The output of the channel is a random variable $Z = X + Y$, where $Y \sim \mathcal{N}(0,1)$. The variables X and Y are independent. Find the probability density function of Z.

3. Consider the following signals defined over $t \in [0,3]$:

$$\psi_1(t) = \begin{cases} \frac{\sqrt{2}}{2} & \text{if } 0 \le t \le 2 \\ 0 & \text{if } 2 < t \le 3 \end{cases} \,, \qquad \psi_2(t) = \begin{cases} 0 & \text{if } 0 \le t \le 2 \\ 1 & \text{if } 2 < t \le 3 \end{cases} \,,$$

   and

$$\psi_3(t) = \begin{cases} \frac{\sqrt{2}}{2} & \text{if } 0 \le t \le 1 \\ -\frac{\sqrt{2}}{2} & \text{if } 1 < t \le 2 \\ 0 & \text{if } 2 < t \le 3 \end{cases} \,.$$

   (a) Show that the signals $\psi_1(t), \psi_2(t), \psi_3(t)$ are orthonormal.

   (b) Express the signal

$$s(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \text{ or } 2 < t \le 3 \\ 1 & \text{if } 1 < t \le 2 \end{cases}$$

   as a weighted linear combination of $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$. Determine the weighting coefficients.

   (c) Use a weighted linear combination

$$\hat{x}(t) = c_1\psi_1(t) + c_2\psi_2(t) + c_3\psi_3(t) \ , \ c_1, c_2, c_3 \in \mathbb{R},$$

   to approximate the function

$$x(t) = t$$

   over the interval $t \in [0,3]$. Find the coefficients c_1, c_2 and c_3 that minimize the mean-square error

$$\text{Err} = \int_0^3 (x(t) - \hat{x}(t))^2 \, dt \,.$$

   What is the value of the approximation error?