MTAT.05.128: Introduction to Digital Communications **University of Tartu**

Final exam

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Student name: ____

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1. This exam contains 10 pages. Check that no pages are missing.
2. It is possible to collect up to 120 points. Try to collect as many points as possible.
3. Justify all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 3 hours.
7. Good luck!

Question 1	
Question 2	
Question 3	
Question 4	
Total	

Question 1 (30 points). Consider the periodic function $f(t)$ of period T given by

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \leq t < T/2 \\ 0 & \text{if} \quad T/2 \leq t < T \end{cases} .$$

(a) Determine its complex Fourier series (you can use any of the three representations).

(b) Determine whether $f(t)$ is a power signal, an energy signal, or neither.

Solution

(a) By using the first representation, we have

$$f(t) = a_0 + \sum_{n=1}^{+\infty} (a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)) ,$$

where

$$a_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{T} \cdot \frac{T}{2} = \frac{1}{2} ,$$

and

$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T f(t) \cos(2\pi n f_0 t) dt = \frac{2}{T} \int_0^{T/2} \cos(2\pi n f_0 t) dt \\ &= \frac{2}{T} \cdot \frac{\sin(2\pi n f_0 t)}{2\pi n f_0} \bigg|_0^{T/2} = \frac{2}{T} \cdot \frac{\sin(\pi n f_0 T)}{2\pi n f_0} - \frac{2}{T} \cdot \frac{\sin(0)}{2\pi n f_0} \\ &= \frac{\sin(\pi n)}{\pi n} - 0 = 0 , \end{aligned}$$

and also

$$\begin{aligned} b_n &= \frac{2}{T} \int_0^T f(t) \sin(2\pi n f_0 t) dt = \frac{2}{T} \int_0^{T/2} \sin(2\pi n f_0 t) dt \\ &= -\frac{2}{T} \cdot \frac{\cos(2\pi n f_0 t)}{2\pi n f_0} \bigg|_0^{T/2} = -\frac{2}{T} \cdot \frac{\cos(\pi n f_0 T)}{2\pi n f_0} + \frac{2}{T} \cdot \frac{\cos(0)}{2\pi n f_0} \\ &= \frac{1 - \cos(\pi n)}{\pi n} = \frac{1 + (-1)^{n+1}}{\pi n} . \end{aligned}$$

To summarize, we obtain

$$f(t) = \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{1 + (-1)^{n+1}}{\pi n} \cdot \sin(2\pi n f_0 t) .$$

(b) The energy of the signal is

$$E = \int_{-\infty}^{+\infty} |f(t)|^2 dt \; .$$

Since the energy of one period

$$E_T = \int_0^T |f(t)|^2 dt = \frac{T}{2}$$

is constant, then $E = +\infty$. It is not an energy signal.

Since the signal is periodic,

$$P = \frac{1}{T} \int_0^T |f(t)|^2 dt = \frac{T}{2T} = \frac{1}{2} \; .$$

It is a power signal.

Question 2 (30 points).

Let $m(t)$ be a message signal, which is bandlimited to the range of $(-w, w)$ Hz. Considered a modulated signal $u(t) = A_c \cdot m(t) \cdot \cos(2\pi f_0 t) \cdot \cos(4\pi f_0 t)$, where A_c is a constant and $w \ll f_0$.

(a) Suggest a demodulation scheme for this modulation and show mathematically that it allows to recover the original signal.

(b) What is the minimum usable value of f_0 (as a function of w)?

Reminder: $\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$.

Solution

(a) By using the formula for product of cosines, the given signal can be rewritten as

$$u(t) = A_c \cdot m(t) \cdot \cos(2\pi f_0 t) \cdot \cos(4\pi f_0 t) = \frac{1}{2} A_c m(t) \cdot (\cos(2\pi f_0 t) + \cos(6\pi f_0 t)) \ .$$

For demodulation, it is suggested to multiply $u(t)$ by $\cos(2\pi f_0 t)$. Then, the resulting signal is:

$$\begin{aligned} u(t)\cos(2\pi f_0 t) &= \frac{1}{2} A_c m(t) \cdot (\cos(2\pi f_0 t) + \cos(6\pi f_0 t)) \cos(2\pi f_0 t) \\ &= \frac{1}{2} A_c m(t) \cos^2(2\pi f_0 t) + \frac{1}{2} A_c m(t) \cos(6\pi f_0 t) \cos(2\pi f_0 t) \\ &= \frac{1}{4} A_c m(t)(1 + \cos(4\pi f_0 t)) + \frac{1}{4} A_c m(t)(\cos(4\pi f_0 t) + \cos(8\pi f_0 t)) \\ &= \frac{1}{4} A_c m(t) + \frac{1}{2} A_c m(t) \cos(4\pi f_0 t) + \frac{1}{4} A_c m(t) \cos(8\pi f_0 t)) \ . \end{aligned}$$

The high frequency components can be removed by the low-pass filter. This leaves us with $\frac{1}{4} A_c m(t)$, which is proportional to the original message signal.

(b) The proposed scheme will work if $2w < 2f_0$, i.e. $w < f_0$.

Question 3 (30 points).

Consider the following two signals defined over $t \in [0, 2]$:

$$\psi_1(t) = \begin{cases} \frac{\sqrt{3}}{2} & \text{if } 0 \le t \le 1 \\ \frac{1}{2} & \text{if } 1 < t \le 2 \end{cases} \qquad \text{and} \qquad \psi_2(t) = \begin{cases} \frac{1}{2} & \text{if } 0 \le t \le 1 \\ -\frac{\sqrt{3}}{2} & \text{if } 1 < t \le 2 \end{cases} .$$

(a) Show that the signals $\psi_1(t)$ and $\psi_2(t)$ are orthonormal.

(b) Express the signal

$$s(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ 1 & \text{if } 1 < t \le 2 \end{cases}$$

as a weighted linear combination of $\psi_1(t)$ and $\psi_2(t)$. Determine the weighting coefficients.

Solution

(a) We have:

$$||\psi_1||^2 = \int_0^2 \psi_1^2(t)dt = \int_0^1 \psi_1^2(t)dt + \int_1^2 \psi_1^2(t)dt = \frac{3}{4} + \frac{1}{4} = 1$$

and

$$||\psi_2||^2 = \int_0^2 \psi_2^2(t)dt = \int_0^1 \psi_2^2(t)dt + \int_1^2 \psi_2^2(t)dt = \frac{1}{4} + \frac{3}{4} = 1.$$

Additionally,

$$\langle \psi_1, \psi_2 \rangle = \int_0^2 \psi_1(t)\psi_2(t)dt = \int_0^1 \psi_1(t)\psi_2(t)dt + \int_1^2 \psi_1(t)\psi_2(t)dt = \frac{3}{4} - \frac{3}{4} = 0 .$$

Therefore, the signals are indeed orthonormal.

(b) Here,

$$\langle s, \psi_1 \rangle = \int_0^2 s(t)\psi_1(t)dt = \int_0^1 s(t)\psi_1(t)dt + \int_1^2 s(t)\psi_1(t)dt = 0 + \frac{1}{2} = \frac{1}{2} ,$$

and

$$\langle s, \psi_2 \rangle = \int_0^2 s(t)\psi_2(t)dt = \int_0^1 s(t)\psi_2(t)dt + \int_1^2 s(t)\psi_2(t)dt = 0 - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} ,$$

Therefore,

$$s(t) = \frac{1}{2}\psi_1(t) - \frac{\sqrt{3}}{2}\psi_2(t) .$$

Question 4 (30 points).

Consider a BPSK-modulated signal transmitted through the AWGN channel with noise spectral density $N_0/2 = \sigma^2$. Let the constellation points of the transmitted signals be $s_1 = -1$ and $s_2 = +1$, and the received signal be r.

In this question we consider the decision error probability when the probabilities of the transmitted signals are

$$\mathsf{Pr}(s_1) = 0.8 \quad \text{and} \quad \mathsf{Pr}(s_2) = 0.2 \ .$$

(a) Assume that the decision regions are chosen according to the following rules:

$$\begin{cases} \text{if } r < 0 \text{ then the decision is } s_1 \\ \text{if } r \geq 0 \text{ then the decision is } s_2 \end{cases} .$$

Show that in this case, similarly to the symmetric case, the probability of error is given by

$$\mathsf{Pr}(\text{err}) = \mathcal{Q}(\sqrt{2E_s/N_0}) \ ,$$

where $\mathcal{Q}(\cdot)$ is the $\mathcal{Q}$-function, and E_r is energy per symbol.

(b) Suggest how to modify the decision regions such that the decision error decreases.

(c) Suggest idea for the optimal choice of the decision regions. There is no need to find the boundary explicitly, but you need to explain how the boundary can be found.

Solution

We have:

(a)

$$\mathsf{Pr}(\text{err}) = \mathsf{Pr}(\text{err}|s_1) \cdot \mathsf{Pr}(s_1) + \mathsf{Pr}(\text{err}|s_2) \cdot \mathsf{Pr}(s_2) \ .$$

Since the decision regions are symmetric, we have

$$\mathsf{Pr}(\text{err}|s_1) = \mathsf{Pr}(\text{err}|s_2) \ ,$$

and so

$$\mathsf{Pr}(\text{err}) = \mathsf{Pr}(\text{err}|s_1) \cdot 0.8 + \mathsf{Pr}(\text{err}|s_1) \cdot 0.2 = \mathsf{Pr}(\text{err}|s_1) \ .$$

Then,

$$\begin{aligned}
\mathsf{Pr}(\text{err}|s_1) &= \int_{r=-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r-s_1)^2}{N_0}\right\} dr \\
&= \int_{r=-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r-\sqrt{E_s})^2}{N_0}\right\} dr \\
&\overset{\text{replace variable}}{=} \int_{x=\sqrt{\frac{2E_s}{N_0}}}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx \\
&= \mathcal{Q}\left(\sqrt{\frac{2E_s}{N_0}}\right) .
\end{aligned}$$

(b) If the decision regions are symmetric, we have

$$\mathsf{Pr}(\text{err}|s_1) = \mathsf{Pr}(\text{err}|s_2)\ .$$

It is given that the signal s_1 is more likely than s_2. If we increase the decision region for s_1 (and decrease the decision region for s_2), then

$$\mathsf{Pr}(\text{err}) = \mathsf{Pr}(\text{err}|s_1) \cdot 0.8 + \mathsf{Pr}(\text{err}|s_2) \cdot 0.2\ . \tag{1}$$

Set the decision boundary to some $b > 0$, namely

$$\begin{cases} \text{if } r < b \text{ then the decision is } s_1 \\ \text{if } r \geq b \text{ then the decision is } s_2 \end{cases} .$$

Now, $\mathsf{Pr}(\text{err}|s_1)$ decreases, and $\mathsf{Pr}(\text{err}|s_2)$ increases. Since $\mathsf{Pr}(\text{err}|s_1)$ is multiplied by higher coefficient, 0.8, the overall error probability decreases.

(c) Since

$$\mathsf{Pr}(s_1) = 0.8 \quad \text{and} \quad \mathsf{Pr}(s_2) = 0.2\ ,$$

the ratio between two probabilities is 4:1. If we gradually increase the boundary value of b, the expression (1) decreases until the break-even point, which is located somewhere between 0 and 1.