MTAT.05.125: Introduction to Theoretical Computer Science **University of Tartu**

Midterm exam: solutions

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Question 1 (48 points).

In the movie festival there are 7 different movies, each movie is screened on a different day.

(a) Robert is a big fan of cinema, so he would like to see all the movies. However, since he is a poor student, his budget allows him to buy only 5 tickets (and this is with the student discount!). Help Robert to count how many ways are there to choose 5 movies out of 7.

Solution. All movies are different and the order we choose them is not important hence the number of ways to choose 5 movies out of 7 is $\binom{7}{5} = 21$.

(b) Martin wants to choose an arbitrary number of movies to see (i.e., Martin can choose to see no movie, or to see any single movie, or any two movies, or any three movies, . . . , or all seven movies). How many choices are there?

Solution. For each movie, Martin can either go or not to go[1] to watch it – there are two options. And for different movies choices are independent. Hence we apply multiplication principle and the answer is $2^7 = 128$.

(c) A group of 14 students wants to see the movies. Each student chooses any number of movies. The choices of students are independent of each other. How many choices are there?

Solution. As we saw in (b), for one student there are 2^7 choices. Since choices of students are independent of each other, we again apply multiplication principle:

$$\left(2^7\right)^{14} = 2^{98}\,.$$

(d) A group of 14 students wants to see the movies, and each movie is seen by exactly two students. How many choices are there?

Solution. For each movie we can choose 2 students out of 14, i.e. in $\binom{14}{2}$ ways. Since choice for one movie is independent from the choice for another movie, we again apply multiplication principle: $\binom{14}{2}^7$.

[1] "To go, or not to go, that is the question...".

(e) A group of 14 students wants to see the movies. Each students chooses 3 movies. Additionally, it is known that each movie is seen by at least one student. How many choices are there?

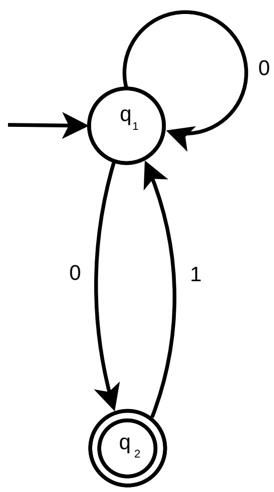
Solution. We use principle of inclusion-exclusion. For $i = 1, 2, \ldots, 7$ define properties P_i = "movie i was not seen by anyone". Then $W(i) = \binom{7}{i}\binom{7-i}{3}^{14}$: we choose i movies out of 7 and do not watch them, other movies can be chosen by students (but we do *not* require each of these movies to be actually chosen); each student choose 3 movies out of $(7-i)$ remaining ones and each student makes his/her choice independently (multiplication principle).

We note that $W(5) = W(6) = W(7) = 0$ therefore the answer is

$$E(0) = \sum_{i=0}^{4}(-1)^i W(i) = \sum_{i=0}^{4}\binom{7}{i}\binom{7-i}{3}^{14}.$$

Question 2 (24 points).

Convert the following nondeterministic finite automaton into equivalent deterministic automaton. Show all the steps in the conversion process.



Solution. We use the standard technique from week 6:

- New state set is a powerset of old states: $P(\{q_1, q_2\}) = \{\varnothing, \{q_1\}, \{q_2\}, \{q_1, q_2\}\} =: \{\bar{q}_\varnothing, \bar{q}_1, \bar{q}_2, \bar{q}_{12}\}$ (we put bars over characters to differentiate old states from the new states).
- Alphabet is the same: $\Sigma = \{0, 1\}$.
- Start stat: $E(q_1) = \bar{q}_1$.
- Set of accept states: $F = \{\bar{q}_2, \bar{q}_{12}\}$ (everything that includes old state q_2).

- Now we start building transition function. We start with the new start state.

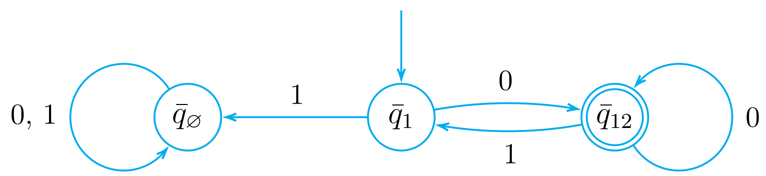
  $\delta(\bar{q}_1, 0) = \bar{q}_{12}$, $\delta(\bar{q}_1, 1) = \bar{q}_\varnothing$. We obtained new reachable states, $\bar{q}_{12}$ and $\bar{q}_\varnothing$.

  $\delta(\bar{q}_{12}, 0) = \bar{q}_{12}$, $\delta(\bar{q}_{12}, 1) = \bar{q}_1$. We did not obtained new reachable states.

  $\delta(\bar{q}_\varnothing, 0) = \delta(\bar{q}_\varnothing, 1) = \bar{q}_\varnothing$. We did not obtain any new reachable states.

  And now we do not have any more reachable states. We did not reach the state $\bar{q}_2$ which means we can ignore it (and, therefore, update $F = \{\bar{q}_{12}\}$).

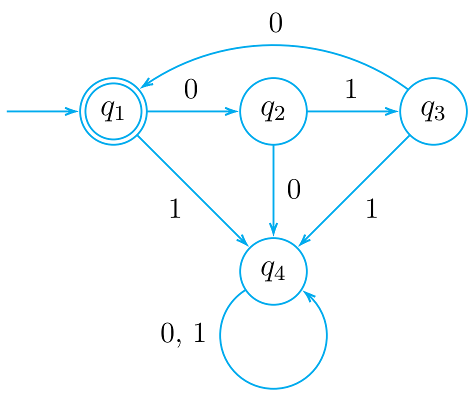
- The state diagram of the DFA:



Question 3 (24 points).

Construct a deterministic finite automaton $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$, which recognizes the language defined by the regular expression: $(010)^*$. Specify all the ingredients Q, Σ, δ, q_0 and F of $\mathcal{M}$.

Solution. The automaton:



This automaton enters from the beginning the accept state q_1 and then moves along the loop $q_1 \to q_2 \to q_3 \to q_1$. Each iteration over the loop corresponds to reading 010, and after each iteration the automaton is in the accept state q_1. If at any step automaton reads not what it expects to follow the loop, it goes to "sink" state q_4 where it stays forever and does not accept.

Set of states of the automaton: $Q = \{q_1, q_2, q_3, q_4\}$, alphabet: $\Sigma = \{0, 1\}$, transition function δ:

	0	1
q_1	q_2	q_4
q_2	q_4	q_3
q_3	q_1	q_4
q_4	q_4	q_4

start state $q_0 = q_1$ and the set of accept states $F = \{q_1\}$.

Note: some of the students forgot about that "sink" state. However, recall that DFA should always have in any state outgoing arrows for all the symbols of the alphabet (otherwise it is NFA).

Question 4 (24 points).

Prove that the following language is not regular:

$$\mathcal{L} = \{\ 0^m 1^n \mid m \leq 2n + 5,\ m, n \in \mathbb{N}\}\ .$$

Solution. We will use the pumping lemma to prove that the language is not regular.

- Assume that $\mathcal{L}$ is regular and p is its pumping length.
- Take the word $w = 0^p 1^p$. Since $p \leq 2p + 5$ then $w \in \mathcal{L}$. Also it is clear that $|w| = 2p \geq p$.
- From pumping lemma we have that $w = xyz$ where x, y and z are such that for all $i \geq 0$ it holds $xy^i z \in \mathcal{L}$. Also $|y| > 0$ and $|xy| \leq p$.
- Since $|xy| \leq p$, both x and y consists of zeros only. Take $i = 2p + 6$ and form the word $xy^{2p+6}z$. According to the pumping lemma this word should belong to $\mathcal{L}$.
- However, $|xy^{2p+6}| \geq (2p + 6)|y| \geq 2p + 6$. It means that the inequality of numbers of zeros and ones defined in $\mathcal{L}$ does not hold any more: $2p + 6 \not\leq 2p + 5$, i.e. $xy^{2p+6}z \notin \mathcal{L}$.
- Contradiction. This means that original assumption was wrong and $\mathcal{L}$ is not regular.