MTAT.05.125: Introduction to Theoretical Computer Science **University of Tartu**

Final exam

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Student name: ____

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1. This exam contains 10 pages. Check that no pages are missing.
2. It is possible to collect up to 120 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 2 hours.
7. Good luck!

Question 1	
Question 2	
Question 3	
Question 4	
Total	

Question 1 (20 points).

Which of the following equations are true or false? Justify your answers.

(a) $n^{10} \cdot \log n = O(n^{10} + n^9)$;

(b) $2^n = o(3^n)$;

(c) $(\log_2 n)^2 = O(\sqrt{n})$.

Question 2 (35 points).

Define the language

$$\mathcal{L}_7 = \left\{ \langle \mathcal{M} \rangle \mid \mathcal{M} \text{ is a Turing machine and } L(\mathcal{M}) = \{0^n 1^{7n} \,|\, n \in \mathbb{N}\} \right\} \ .$$

In this question, you will show that $\mathcal{L}_7$ is an undecidable language.

Hint: for example, you can use a reduction from the language $\mathcal{L}_{\text{TM}}$. Assume that there exists a Turing machine $\mathcal{M}_7$ that decides $\mathcal{L}_7$. Construct a Turing machine $\mathcal{M}_{\text{TM}}$ that decides $\mathcal{L}_{\text{TM}}$, where

$$\mathcal{L}_{\text{TM}} = \{\langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a Turing machine and } \mathcal{M} \text{ accepts the input string } w\} \ .$$

On the input $\langle \mathcal{M}, w \rangle$, the machine $\mathcal{M}_{\text{TM}}$ does the following:

1. Constructs a machine $\mathcal{M}_w$, which on the input x does the following:
   (a) "simulates" the run of $\mathcal{M}$ on w;
   (b) if $\mathcal{M}$ rejects, – $\mathcal{M}_w$ rejects;
   (c) if $\mathcal{M}$ accepts, – $\mathcal{M}_w$ checks whether x has the form $0^n 1^{7n}$, $n \in \mathbb{N}$. If yes, – accepts. If no, – rejects.
2. Uses $\mathcal{M}_7$ to determine whether $L(\mathcal{M}_w) = \{0^n 1^{7n}\}$. If yes, – accepts. If no, – rejects.

Complete the details of the reduction if needed, and show that $\mathcal{L}_7$ is an undecidable language.

Question 3 (25 points).

Definition: a *center* in an undirected graph $\mathcal{G}$ is a vertex v such that for every vertex u in $\mathcal{G}$, $v \neq u$, there is an undirected edge between u and v. (In other words, a center is a vertex that connected to all other vertices.)

Define a language HASCENTER:

$$\mathsf{HASCENTER} = \{\langle \mathcal{G}, v \rangle \mid \mathcal{G} \text{ is an undirected graph with center } v \ \} \ .$$

Is $\mathsf{HASCENTER} \in \mathcal{P}$? Justify your answer.

Question 4 (40 points).

Definition: an *independent set* in an undirected graph $\mathcal{G}$ is a set of vertices S such that for every two vertices u, v in S there is no edge connecting between u and v.

Define a language $\mathsf{INDEPENDENT\text{-}SET}$:

$$\mathsf{INDEPENDENT\text{-}SET} = \{\langle \mathcal{G}, k \rangle \mid \mathcal{G} \text{ is an undirected graph with an independent set of size } k\} \ .$$

In this question, you will show that $\mathsf{INDEPENDENT\text{-}SET}$ is $\mathcal{NP}$-complete.

(a) Prove that $\mathsf{INDEPENDENT\text{-}SET} \in \mathcal{NP}$.

(b) Prove that $\mathsf{INDEPENDENT\text{-}SET}$ is $\mathcal{NP}$-hard.

Hint: you can use a polynomial-time reduction from CLIQUE to $\mathsf{INDEPENDENT\text{-}SET}$. Recall that a language CLIQUE is defined as follows:

$$\mathsf{CLIQUE} = \{\langle \mathcal{G}, k \rangle \mid \mathcal{G} \text{ is an undirected graph with a clique of size } k\} \ .$$

It is known (it was shown in the class) that CLIQUE is $\mathcal{NP}$-complete.