1. This exam contains 10 pages. Check that no pages are missing.

2. It is possible to collect up to 110 points. Try to collect as many points as possible.

3. Justify and prove all your answers (where applicable).

4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.

5. Any printed and written material is allowed in the class. No electronic devices are allowed.

6. Exam duration is 2 hours.

7. Good luck!
Question 1 (30 points).

Define the language

\[ \mathcal{L}_1 = \{ \langle A \rangle \mid A \text{ is a DFA, and all strings of the form } 10^n1, n \geq 1, \text{ are accepted by } A \} . \]

Show that \( \mathcal{L}_1 \) is a decidable language.

**Hint:** note that \( A \) can also accept strings, which are not of the form \( 10^n1 \). One possible way to solve this question is to use the fact that \( \mathcal{L}_\emptyset \) is a decidable language (shown in the lecture), where

\[ \mathcal{L}_\emptyset = \{ \langle A \rangle \mid A \text{ is a DFA and } L(A) = \emptyset \} . \]
Question 2 (30 points).

Define the language

\[ L_2 = \{ \langle M \rangle \mid M \text{ is a Turing machine that accepts some, but not all, strings of the form } 0^n1^n, n \geq 1 \} . \]

In this question, you will show that \( L_2 \) is an undecidable language.

**Hint:** for example, you can use a reduction from the language \( L_{TM} \). Assume that there exists a Turing machine \( M_2 \) that decides \( L_2 \). Construct a Turing machine \( M_{TM} \) that decides \( L_{TM} \), where

\[ L_{TM} = \{ \langle M, w \rangle \mid M \text{ is a Turing machine and } M \text{ accepts the input string } w \} . \]
Define a language \textsc{sum-eight-numbers}:

\[
\textsc{sum-eight-numbers} = \left\{ \langle A, k \rangle \mid A \text{ is an array of integer numbers} \right. \\
\left. \text{and } k \text{ is a sum of eight different elements from } A \right\}.
\]

Is \textsc{sum-eight-numbers} \in \mathcal{P}? Justify your answer.
Question 4 (30 points).

**Definition**: a bipartite clique in an undirected finite graph $G(V, E)$ is a subgraph with a set of vertices $A \cup B$, such that $A$ and $B$ are disjoint, the number of vertices in $A$ and in $B$ are equal, and every vertex $v \in A$ is connected to every vertex $u \in B$ with an edge from $E$. The size of the bipartite clique is the number of vertices in $A$ (or, equivalently, in $B$).

Define a language $\text{BIPARTITE-CLIQUE}$:

$$\text{BIPARTITE-CLIQUE} = \{ \langle G, k \rangle \mid G \text{ is an undirected graph that has a bipartite clique of size } k \}.$$

In this question, you will show that $\text{BIPARTITE-CLIQUE}$ is $\mathcal{NP}$-complete.

(a) Prove that $\text{BIPARTITE-CLIQUE} \in \mathcal{NP}$.

(b) Prove that $\text{BIPARTITE-CLIQUE}$ is $\mathcal{NP}$-hard.

**Hint**: you can use a polynomial-time reduction from $\text{CLIQUE}$ to $\text{BIPARTITE-CLIQUE}$. Do not forget to show that the reduction is correct and polynomial-time.