MTAT.03.286: Design and Analysis of Algorithms **University of Tartu**

Midterm exam

November 11th, 2019

Student name: ____

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1. This exam contains 8 pages. Check that no pages are missing.
2. It is possible to collect up to 120 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable). Show all important steps in your solution.
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 1 hour 40 minutes.
7. Good luck!

Question 1	
Question 2	
Question 3	
Total	

Question 1 (40 points). Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a finite connected undirected graph without self-loops. Let $w : \mathcal{E} \mapsto \mathbb{R}$ be a weight function, such that $w(e) > 0$ for all edges $e \in \mathcal{E}$. Additionally, it is known that the weights of all edges in $\mathcal{G}$ are different.

Prove or disprove the following statements.

(a) If $\mathcal{T}(\mathcal{V}, \mathcal{E}_T)$ is a minimum spanning tree of $\mathcal{G}$, and e is the edge of the *minimum* weight in $\mathcal{G}$, then it holds that $e \in \mathcal{E}_T$.

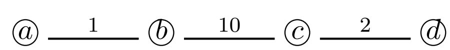
(b) If $\mathcal{T}(\mathcal{V}, \mathcal{E}_T)$ is a minimum spanning tree of $\mathcal{G}$, and e is the edge of the *maximum* weight in $\mathcal{G}$, then it holds that $e \notin \mathcal{E}_T$.

Solution:

(a) The statement is correct.

Proof. By contrary, consider a minimum spanning tree of $\mathcal{G}$, $\mathcal{T}(\mathcal{V}, \mathcal{E}_T)$, such that $e \notin \mathcal{E}_T$. If we add the edge e to $\mathcal{T}$, then a simple circuit is formed, denote it $\mathcal{C}$. Remove any edge e', $e' \neq e$, from $\mathcal{C}$. The resulting subgraph $\mathcal{T}$ with e' replaced by e is a spanning tree because it is connected, and the number of edges in it is the same as in $\mathcal{T}$, namely $|\mathcal{V}| - 1$. Since the weight of e is minimum, the weight of the spanning tree obtained by replacing e' by e is smaller than that of $\mathcal{T}$, by contradiction to its minimality. $\square$

(b) The statement is not correct. We show this by the following counterexample. Take $\mathcal{V} = \{a, b, c, d\}$, and the edges as shown below:



Here $\{b, c\}$ is the edge of the maximum weight 10 in $\mathcal{G}$. A path that connects vertices b and c must use the edge $\{b, c\}$. Therefore, any spanning tree contains it.

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Question 2 (50 points). Let $\mathcal{N}(\mathcal{G}, s, t, c)$ be a flow network, where $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a finite directed graph. Prove that there exists a *maximum* flow function in $\mathcal{N}$ that does not contain a cycle.

Defintion: Let $f : \mathcal{E} \mapsto \mathbb{R}$ be a non-negative real-valued flow function in $\mathcal{N}$. We say that f *contains a cycle* if there exists a simple circuit in $\mathcal{G}$, such that for every edge e in this circuit $f(e) > 0$.

Solution:

By contrary, assume that there exists a network $\mathcal{N}(\mathcal{G}, s, t, c)$, which has no maximum flow that does not contain a simple circuit. Take a maximum flow function f *with the smallest possible number of simple circuits* (where each edge e in every such circuit has $f(e) > 0$).

Pick one such simple circuit $\mathcal{C}$. Denote by Δ the smallest value of $f(e)$ among all edges $e \in \mathcal{C}$. Define a new flow function f' as follows:

$$f'(e) = \begin{cases} f(e) - \Delta & \text{if } e \in \mathcal{C} \\ f(e) & \text{otherwise} \end{cases} .$$

In other words, we substract Δ units of flow from each edge participating in the simple circuit $\mathcal{C}$.

The edge rule is preserved. Indeed, we do not change the flow in the edges, which are not in $\mathcal{C}$. For the edges, which are in $\mathcal{C}$, we decrease the flow by Δ, but this flow was initially at least Δ. Therefore, for $e \in \mathcal{C}$,

$$c(e) \geq f(e) \geq f'(e) = f(e) - \Delta \geq \Delta - \Delta = 0 ,$$

and therefore the edge rule is preserved.

The vertex rule is preserved.

- For any vertex, which is not in $\mathcal{C}$, we do not change any flows in the edges incident with it. So the vertex rule is preserved.
- For any vertex, which is in $\mathcal{C}$, each time we meet this vertex in the circuit, we decrease the incoming and outgoing flows by Δ, and therefore the total incoming and outgoing flows remain equal.

The flow function f' is maximum. If the vertex t does not participate in $\mathcal{C}$, then the incoming flow into t and the outgoing flow out of t is not changed. If $t \in \mathcal{C}$, then each time we meet this vertex in the circuit, we decrease the incoming and outgoing flows by Δ, and therefore the difference between the total incoming flow into t and the total outgoing flow from t remains unchanged. In other words,

$$\sum_{e \in \text{In}(t)} f'(e) - \sum_{e \in \text{Out}(t)} f'(e) = \sum_{e \in \text{In}(t)} f(e) - \sum_{e \in \text{Out}(t)} f(e) ,$$

and therefore the total flow for f' is maximum due to the maximality of the flow f.

The number of circuits in f' is decreased by one. Indeed, Δ was taken as the (smallest) value of $f(e)$ for some $e \in \mathcal{C}$. Then, $f'(e) = 0$, and therefore the circuit $\mathcal{C}$ does not exist in the flow function f'. Since we reduced the flows in edges, no new simple circuits in the flow were created. This is in contradiction to the assumption that f has the smallest possible number of circuits. $\square$

Alternative solution

It is also possible to take a flow function f with k simple circuits, and by using a similar technique to show that there exists a flow function f' with $\leq k-1$ simple circuits. By repeating this argument up to k times, we obtain a flow function without simple circuits.

Question 3 (30 points).

By using the Fast Fourier Transform (FFT) algorithm, evaluate the polynomial $P(x) = 2x^3 - x^2 - 4x + 6$ at the complex 4th roots of unity. Show at least one level of recursion.

We choose the 4th roots of unity so that $\omega^4 = 1$ and $\omega = i$. Hence $\omega^2 = -1$, $\omega^3 = -i$.

At first we write out the FFT matrix and then we rearrange the columns.

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^3 \\ 1 & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^3 & \omega \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \\ -4 \\ 2 \end{pmatrix} =$$

$$= \begin{pmatrix} M_2(\omega^2) & E \cdot M_2(\omega^2) \\ M_2(\omega^2) & F \cdot M_2(\omega^2) \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \\ -4 \\ 2 \end{pmatrix}$$

Here

$$E = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \qquad F = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -\omega \end{pmatrix} = -E$$

We now recursively evaluate

$$M_2(\omega^2) \cdot \begin{pmatrix} 6 \\ -1 \end{pmatrix} \qquad M_2(\omega^2) \cdot \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

We can write this out using FFT to reduce this to multiplying with $M_1 = 1$.

$$M_2(\omega^2) \cdot \begin{pmatrix} 6 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$M_2(\omega^2) \cdot \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$$

Now, we can put this together

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -6 \end{pmatrix} \\ \begin{pmatrix} 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -6 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 5 - 2 \\ 7 - 6\omega \\ 5 + 2 \\ 7 + 6\omega \end{pmatrix} = \begin{pmatrix} 3 \\ 7 - 6i \\ 7 \\ 7 + 6i \end{pmatrix}$$

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