1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of variables $x_i \in \{-1, +1\}, n \geq 1$. Consider an inequality $\phi$ with $m \geq 1$ different variables of the form
\[ \pm x_{t_1} \pm x_{t_2} \pm \cdots \pm x_{t_m} \geq 0, \]
and $t_i \in \{1, 2, \ldots, n\}$ for all $i = 1, 2, \ldots, m$ (here, ‘$\pm$’ denotes a sign which can be either ‘$+$’ or ‘$-$’).

(a) Consider a simple probabilistic algorithm that assigns values $-1$ and $+1$ to all $x_i, i = 1, 2, \ldots, n$, as follows
\[ \Pr[x_i = -1] = \Pr[x_i = +1] = \frac{1}{2} \]
(the value to each variable is assigned independently of other variables). Show that the probability that the proposed algorithm does not find a solution of $\phi$ is at most $\frac{1}{2}$.

Hint: Show that for each assignment of values to $x_i$ that does not solve $\phi$ there is an assignment that solves $\phi$.

(b) Consider now a system of equations $\phi_1, \phi_2, \ldots, \phi_r$ as above, $r \geq 1$, where any two equations $\phi_i$ and $\phi_j$ use a disjoint set of unknowns. What is the probability that the algorithm in part (a) solves the system in (b) (i.e., all $r$ equations are solved at the same time)?

(c) How many times should the algorithm in (a) be executed in order to solve the system given in (b) with probability at least $\frac{1}{2}$? with probability at least $1 - \epsilon$, where $\epsilon > 0$ is some constant?

2. Convert the following linear-programming problem into an equivalent canonical form:
\[
\begin{align*}
\text{min} & \quad x_1 + 2x_2 - x_3 \\
\text{s.t.} & \quad 2x_1 - x_2 + x_3 \leq 3 \\
& \quad 5x_1 + 3x_2 - x_3 \geq 2 \\
& \quad x_1 - 2x_3 = 5 \\
& \quad x_2 \geq 0, \quad x_3 \leq 0
\end{align*}
\]

Reminder: a linear-programming problem is said to be in a canonical form if the following conditions hold:

- the objective function is a maximization;
3. Write the dual of the following linear-programming problem:

\[
\begin{align*}
\text{max} & \quad x_1 - 3x_2 - 2x_3 \\
\text{s.t.} & \quad 2x_1 - x_2 + x_3 \leq 4 \\
& \quad 3x_2 - x_3 = 2 \\
& \quad x_1 + 2x_3 \leq 1 \\
& \quad x_1 \geq 0, \quad x_3 \geq 0
\end{align*}
\]

(in this question there is no need to find the optimum of the given problem.)

4. Consider the following flow network \( \mathcal{N} \).

(a) Write the problem of finding maximum flow from \( s \) to \( t \) in \( \mathcal{N} \) as a linear program.

(b) Write down the dual of this linear program. There should be a dual variable for each edge of the network and for each vertex other than \( s \) and \( t \).

Now, consider a general flow network. Recall the linear program formulation for a general maximum flow problem, which was shown in the class.

(c) Write down the dual of this general flow linear-programming problem, using a variable \( y_e \) for each edge and \( x_v \) for each vertex \( v \neq s, t \).

(d) Show that any solution to the general dual problem must satisfy the following property: for any directed path from \( s \) to \( t \) in the network, the sum of \( y_e \) values along the path must be at least 1.