MTAT.03.286: Design and Analysis of Algorithms **University of Tartu**

Final exam

January 13th, 2020

Student name: ____

Student ID: ____

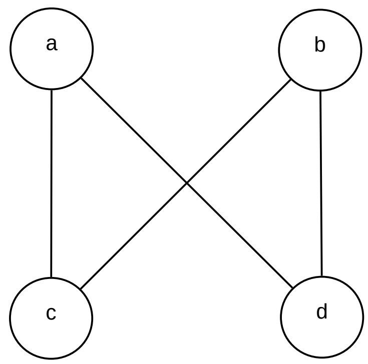
1. This exam contains 8 pages. Check that no pages are missing.
2. It is possible to collect up to 120 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 1 hour and 40 minutes.
7. Good luck!

Question 1	
Question 2	
Question 3	
Total	

Question 1 (35 points).

Reminder: A *vertex cover* in an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a subset of vertices $\mathcal{S} \subseteq \mathcal{V}$ such that every edge $e \in \mathcal{E}$ is incident with at least one vertex in $\mathcal{S}$. The *size* of $\mathcal{S}$ is a number of vertices in it.

We are interested in finding a minimum vertex cover $\mathcal{S}$ in the following undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$:



(a) For each vertex $v \in \mathcal{V}$ define an indicator variable $\mathcal{X}_v$ ($v \in \{a, b, c, d\}$):

$$\mathcal{X}_v = \begin{cases} 1 & \text{if } v \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases} .$$

Formulate a problem of finding a minimum vertex cover in $\mathcal{G}$ as an *integer linear programming (ILP)* problem.

(b) Relax the ILP problem in part (a) to become a *linear programming (LP)* problem (explain the steps in your solution).

(c) What is an optimum solution to the ILP problem in part (a)?

(d) Does the choice $\mathcal{X}_a = \mathcal{X}_b = \mathcal{X}_c = \mathcal{X}_d = \frac{1}{2}$ give a feasible solution to the LP problem in (b)? Is it an optimum solution? Justify your answer.

(e) Does the choice in part (d) give an extreme point solution of the problem in part (b)?

Question 2 (45 points). You are given the following linear-programming (LP) problem:

$$\begin{array}{rrcrcrcr} \textbf{max} & x_1 & + & x_2 & - & 2x_3 & & \\ \textbf{s.t.} & x_1 & + & x_2 & & & \leq & 5 \\ & x_1 & & & + & x_3 & \leq & 3 \\ - & x_1 & + & x_2 & + & 2x_3 & \leq & 0 \\ & \multicolumn{7}{l}{x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0} \end{array}$$

(a) Solve this LP problem by using the simplex method.

(b) Formulate the dual LP problem.

(c) Verify that your solution of (a) is optimal by the substitution of the point $(1, 0, 0)$ into the dual problem. Explain your answer.

Question 3 (40 points).

Reminder:

- An *edge cover* in an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a subset of edges $\mathcal{C} \subseteq \mathcal{E}$ such that any vertex $v \in \mathcal{V}$ is incident with at least one edge in $\mathcal{C}$. The *size* of the edge cover is a number of edges in it.
- An *independent set* in an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a subset of vertices $\mathcal{I} \subseteq \mathcal{V}$ such that no two vertices in $\mathcal{I}$ are connected by an edge. The *size* of the independent set is a number of vertices in it.

Example:

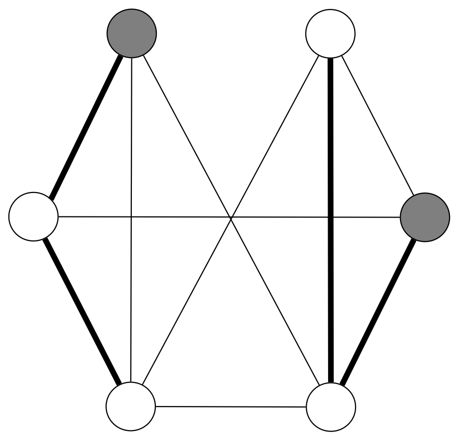


Figure 1: The bold edges form an edge cover. This edge cover has size 4, while the smallest edge cover in this graph has size 3. Moreover, the two grey vertices form an independent set.

Assume that $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is an undirected finite graph with degree of every vertex $v \in \mathcal{V}$ equal 3. Propose a simple algorithm for finding a smallest edge cover in $\mathcal{G}$ with approximation factor 3. Prove correctness of your solution.

Hint: it is possible to solve this problem by finding a maximal independent set first.