Introduction into Linear Programming

Example: chocolate factory

Consider a chocolate factory that produces two types of chocolate boxes.

Type I: €1 profit per box;
Type II: €3 profit per box.

Assume that the factory produces $x_1$ boxes of Type I and $x_2$ boxes of Type II per day. Assume also that the maximum production capacity of the factory, when it produces only boxes of Type I, is 300 boxes per day, and that the maximum production capacity, when it produces only boxes of Type II, is 200 boxes per day. If the factory produces two types of boxes, then we assume that $x_1 + x_2 \leq 400$. The goal is to maximize the profit of the factory.

The profit of the factory in euro is given by the expression $x_1 + 3x_2$. This will be called the objective function. The constraints are the following inequalities:

$$x_1 \leq 300$$
$$x_2 \leq 200$$
$$x_1 + x_2 \leq 400$$

Overall, the problem can be written in the following form:

$$\text{max } x_1 + 3x_2$$
$$\text{s.t. } x_1 \leq 300$$
$$x_2 \leq 200$$
$$x_1 + x_2 \leq 400$$
$$x_1 \geq 0, \ x_2 \geq 0$$

where we added the requirement that $x_1$ and $x_2$ are non-negative. The space of all possible combinations $(x_1, x_2)$ that represent the daily output of the factory is shown in the following picture:
This region is called the \textit{feasible region} of the problem. The blue lines represent the combinations \((x_1, x_2)\) for which the profit is the same (i.e., two points are on the same blue line if the bring the same profit to the factory). As we move upwards, the profit increases. It can be seen that the maximum profit is obtained at the vertex of the feasible region. This is a general property of linear programming problems, as we show below, except for the two cases:

- There is no feasible points. For example, take constraints \(x_1 \leq 0\) and \(x_1 \geq 2\).
- The feasible region is unbounded. For example, take the objective function \(x_1 + x_2\) together with the constraints \(x_1 \geq 0\) and \(x_2 \geq 0\).

\textbf{Variants of linear programming}

1. The objective function can be either maximization or minimization. We can switch between the two by multiplying all the coefficients in the objective function by \(-1\).

2. (a) To turn inequality into equality, we add so-called \textit{slack variables}. For example,

\[
\sum_{i=1}^{n} a_ix_i \leq b
\]

becomes

\[
\sum_{i=1}^{n} a_ix_i + s = b, \text{ where } s \geq 0.
\]
(b) To turn an equality into inequality, we simply use two inequalities as follows:

\[
\sum_{i=1}^{n} a_i x_i = b
\]

becomes

\[
\sum_{i=1}^{n} a_i x_i \geq b,
\]

\[
\sum_{i=1}^{n} a_i x_i \leq b.
\]

3. Variable \( x \) which is unrestricted (i.e. can be either positive or negative) can be represented in the following way:

\[
x = x^+ - x^-,
\]

\[
x^+, x^- \geq 0.
\]

Matrix-vector representation

The general representation of the linear programming problem in the canonical form is when the objective function is maximization, all constraints are inequalities “\( \leq \)” and all variables are non-negative. It can be written in the following form:

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \forall i : \sum_{j=1}^{n} a_{ij} x_j \leq b_i \\
& \quad \forall j : x_j \geq 0.
\end{align*}
\]

Next, denote as follows:

\[
\begin{align*}
b &= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, &
\quad c &= \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, &
\quad x &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, &
\quad A &= (a_{ij})_{i=1,\ldots,m}^{j=1,\ldots,n}.
\end{align*}
\]

Then, the last problem can be re-written in the matrix-vector form as follows:

\[
\begin{align*}
\text{max} & \quad c^T \cdot x \\
\text{s.t.} & \quad A x \leq b \\
& \quad x \geq 0.
\end{align*}
\]

Convexity of the feasible region

**Definition 1** The region \( \mathcal{R} \) is called convex if for any two points \( x, y \in \mathcal{R} \) it holds that

\[
\alpha x + (1 - \alpha) y \in \mathcal{R} \quad \text{for all} \quad \alpha \in (0, 1).
\]
Consider two feasible points \( x \) and \( y \) of some linear programming problem. The two points satisfy the constraints \( Ax \leq b \) and \( Ay \leq b \) of the given problem. Then,

\[
A(\alpha x + (1 - \alpha)y) = \alpha Ax + (1 - \alpha)Ay \leq \alpha b + (1 - \alpha)b = b.
\]

Therefore, \( \alpha x + (1 - \alpha)y \) is also a feasible point. Therefore, the feasible region is convex.

**Optimum is obtained at the vertex**

Consider a bounded linear-programming problem, which has a feasible solution. We show that the optimum is obtained at the boundary of the feasible region.

Consider a canonical form of the linear-programming (LP) problem. Let \( p \) be the optimum point, which is not on the boundary of the feasible region. Without loss of optimality assume that the objective function is maximization. Take \( x \) and \( y \) to be points on the boundary such that

\[
p = \alpha x + (1 - \alpha)y, \quad \alpha \in (0, 1).
\]

We have

\[
c^T p = c^T(\alpha x + (1 - \alpha)y) = \alpha c^T x + (1 - \alpha)c^T y \leq \alpha c^T p + (1 - \alpha)c^T p = c^T p,
\]

where the "\( \leq \)" transition is due to the fact that \( p \) maximizes the objective function.

Since we started and ended up with the same value \( c^T p \), all transitions must be equalities. This is possible only if \( c^T p = c^T x = c^T y \), and therefore \( x \) and \( y \) are also optimum points.

We can continue this approach until we end up with a vertex of the feasible region.

**Example: three-dimensional case**

Consider a region defined by three inequalities.
Facet: when one of the inequalities is replaced with equality.

Edge: two of the inequalities are replaced with equalities.

Vertex: three of the inequalities are replaced with equalities.

Dual problem

Consider again a matrix-vector representation of the LP problem in a canonical form:

\[
\begin{align*}
\max & \quad c^T \cdot x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0.
\end{align*}
\]

The dual problem is defined as follows (while the original problem is called primal):

\[
\begin{align*}
\min & \quad y^T \cdot b \\
\text{s.t.} & \quad y^T A \geq c^T \\
& \quad y \geq 0.
\end{align*}
\]

Weak duality theorem

The following theorem is called a weak duality theorem.

**Theorem 1** Primal maximum \( \leq \) dual minimum.

**Proof.** We show that if \( x \) is any primal feasible point and \( y \) is any dual feasible point then

\[
c^T x \leq y^T b.
\]

We have \( Ax \leq b \). Since \( y \geq 0 \), \( y^T A x \leq y^T b \). Since \( x \geq 0 \), \( y^T A \geq c^T \), we have \( y^T A x \geq c^T x \).

By putting things together, we have \( y^T b \geq c^T x \) for any two feasible points \( x \) and \( y \) of primal and dual LP problem, respectively. In particular, this holds for the maximum of \( c^T x \) and for minimum of \( y^T b \). \( \square \)

Certificate of optimality

If \( x \) and \( y \) are feasible solutions to the primal and dual problem, respectively, and \( c^T x = y^T b \), then \( x \) and \( y \) must be optimal solutions to the primal and dual LP problems, respectively.