It is possible to collect up to 110 points in this homework assignment.

1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of binary variables of size $n \geq 1$, $x_i \in \{0, 1, 2, 3\}$. Consider an equation $\phi$ with $m \geq 1$ unknowns of the form $x_{t_1} + x_{t_2} + \cdots + x_{t_m} = 0$, where all computations are done modulo 4, and $t_i \in \{1, 2, \ldots, n\}$ for all $i = 1, 2, \ldots, m$.

   (a) Consider a simple probabilistic algorithm that assigns values 0, 1, 2 and 3 to all $x_i$, $i = 1, 2, \ldots, n$, as follows
   \[
   \Pr[x_i = 0] = \Pr[x_i = 1] = \Pr[x_i = 2] = \Pr[x_i = 3] = \frac{1}{4}
   \]
   (the value to each variable is assigned independently of other variables). Show that the probability that the proposed algorithm does not find a solution of $\phi$ is exactly $\frac{3}{4}$.

   Hint: choose the values for $x_{t_1}, x_{t_2}, \ldots, x_{t_{m-1}}$. Then, for each assignment of value to $x_{t_m}$ that solves $\phi$ there are three assignments that do not solve $\phi$.

   (b) Consider now a system of equations $\phi_1, \phi_2, \ldots, \phi_r$ as above, $r \geq 1$, where any two equations $\phi_i$ and $\phi_j$ use a disjoint set of unknowns. What is the probability that the algorithm in part (a) solves the system in (b) (i.e., all $r$ equations are solved at the same time)?

   (c) How many times should the algorithm in (a) be executed in order to solve the system given in (b) with probability at least $\frac{1}{2}$? with probability at least $1 - \epsilon$, where $\epsilon > 0$ is some constant?

2. Write the dual of the following linear-programming problem:

   \[
   \begin{align*}
   \text{max} & \quad 2x_1 + x_2 + 3x_3 \\
   \text{s.t.} & \quad 2x_1 + x_2 \leq 3 \\
   & \quad 3x_2 + x_3 \leq 4 \\
   & \quad x_3 \leq 1 \\
   & \quad x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0
   \end{align*}
   \]

   (in this question there is no need to find the optimum of the given problem.)
3. Consider the following flow network $N$.

![Flow Network Diagram]

(a) Write the problem of finding maximum flow from $s$ to $t$ in $N$ as a linear program.

(b) Write down the dual of this linear program. There should be a dual variable for each edge of the network and for each vertex other than $s$ and $t$.

Now, consider a general flow network. Recall the linear program formulation for a general maximum flow problem, which was shown in the class.

(c) Write down the dual of this general flow linear-programming problem, using a variable $y_e$ for each edge and $x_v$ for each vertex $v \neq s, t$.

(d) Show that any solution to the general dual problem must satisfy the following property: for any directed path from $s$ to $t$ in the network, the sum of $y_e$ values along the path must be at least 1.

4. Definition. An edge cover of a graph is a set of edges such that every vertex of the graph is incident with at least one edge of the set. The minimum edge cover problem is the problem of finding an edge cover of minimum size.

Definition. An independent set is a set of vertices in a graph, no two of which are adjacent. The maximum independent set problem is the problem of finding an independent set of maximum size.

(a) Formulate minimum edge cover problem as integer linear-programming problem.

(b) Formulate maximum independent set problem as integer linear-programming problem.

(c) In the two formulations, relax constraints such that the resulting problems become linear-programming problems, and the resulting two problems are dual.