MTAT.03.286: Advanced Methods in Algorithms **University of Tartu**

Final exam

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Student name: ____

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1. This exam contains 10 pages. Check that no pages are missing.
2. It is possible to collect up to 110 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 3 hours.
7. Good luck!

Question 1	
Question 2	
Question 3	
Question 4	
Total	

Question 1 (20 points).

By using Fast Fourier Transform (FFT) algorithm, evaluate the polynomial $P(x) = x^3 - x^2 + 5x - 3$ at the complex 4th roots of unity. Show at least one level of recursion.

Question 2 (25 points). Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a finite connected undirected graph with the real-valued weight function w defined on the edges, such that $w(e) > 0$ for all edges $e \in \mathcal{E}$. Assume that $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ is a *connected* subgraph of $\mathcal{G}$ (in particular, $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$).

Prove or disprove the following statements.

(a) If $\mathcal{T}(\mathcal{V}, \mathcal{E}_T)$ is a minimum spanning tree of $\mathcal{G}$ then the set of edges $\mathcal{E}_T \cap \mathcal{E}'$ defines a minimum spanning tree of $\mathcal{G}'$.

(b) If $\mathcal{T}'(\mathcal{V}', \mathcal{E}_S)$ is a minimum spanning tree of $\mathcal{G}'$, then there exists a spanning tree $\mathcal{T}(\mathcal{V}, \mathcal{E}_T)$ of $\mathcal{G}$ such that $\mathcal{E}_S \subseteq \mathcal{E}_T$ and $\mathcal{T}$ is a *minimum* spanning tree.

Question 3 (30 points). A flow network *with finite capacities of vertices* is defined as a 5-tuple $\mathcal{N}(\mathcal{G}(\mathcal{V},\mathcal{E}),s,t,b,c)$, such that $\mathcal{N}(\mathcal{G}(\mathcal{V},\mathcal{E}),s,t,c)$ is a standard flow network. Additionally, $b:\mathcal{V}\to\mathbb{Q}^+$ is a vertex capacity function, such that any flow function $f:\mathcal{E}\to\mathbb{R}$ in $\mathcal{N}$ must satisfy

$$\sum_{e\in\mathrm{in}(v)} f(e) \le b(v)$$

for any vertex $v\in\mathcal{V}\setminus\{s,t\}$.

Propose an efficient algorithm which finds a maximum flow in a network with finite capacities of vertices with time complexity $O(|\mathcal{V}|^2|\mathcal{E}|)$ and prove its correctness (you can assume that for any vertex $v\in\mathcal{V}$ there is a path from s to v and a path from v to t in $\mathcal{G}$).

Question 4 (35 points).

Definitions:

1. *An independent set* in an undirected finite graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is a subset of vertices $\mathcal{C} \subseteq \mathcal{V}$, such that no two vertices in $\mathcal{C}$ are adjacent.
2. Given an undirected finite graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, *a maximum independent set problem* is a problem of finding an independent set of maximum size.
3. *An extreme point solution* is a feasible solution that cannot be expressed as a convex combination of other feasible solutions.
4. *A half-integral solution* is a feasible solution in which each variable is either 0, 1 or $\frac{1}{2}$. *An integral solution* is a feasible solution in which each variable is either 0 or 1.

We formulate the following relaxation of an integer linear programming problem for a maximum independent set problem:

$$\begin{aligned} \textsf{maximize} \quad & \sum_{v \in \mathcal{V}} x_v \\ \textsf{subject to} \quad & \forall e = \{u,v\} \in \mathcal{E} \; : \; x_u + x_v \leq 1 \\ & \forall v \in \mathcal{V} \; : \; x_v \geq 0 \end{aligned}$$

Here, for all $v \in \mathcal{V}$, we used indicator variables x_v, which are 1 if $v \in \mathcal{C}$, and 0 otherwise.

(a) Show that any feasible solution, which is not half-integral, is not an extreme point solution.

(b) By using the result in (a), show that if the graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is bipartite then all extreme point solutions are integral.

(c) Assume that the graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is bipartite. Propose a linear-programming based algorithm for solving a maximum independent set problem in $\mathcal{G}$. Justify your solution.