MTAT.03.286: Design and Analysis of Algorithms **University of Tartu**

Final exam

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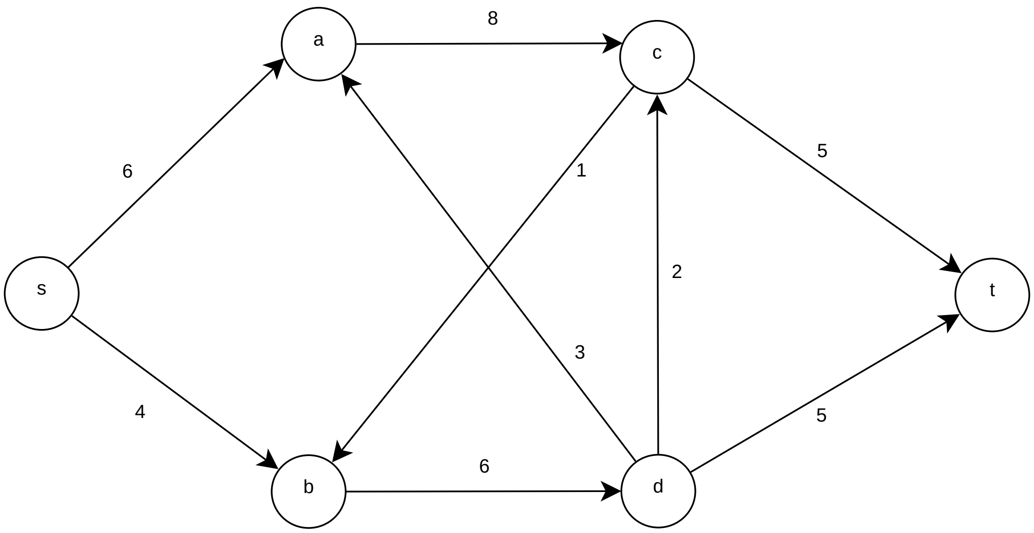
Student name: ______________________________________

Student ID: ____

1. This exam contains 8 pages. Check that no pages are missing.
2. It is possible to collect up to 110 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 1 hour and 40 minutes.
7. Good luck!

Question 1	
Question 2	
Question 3	
Total	

Question 1 (25 points). Find a maximum flow between s and t in the following network by using Dinitz algorithm:

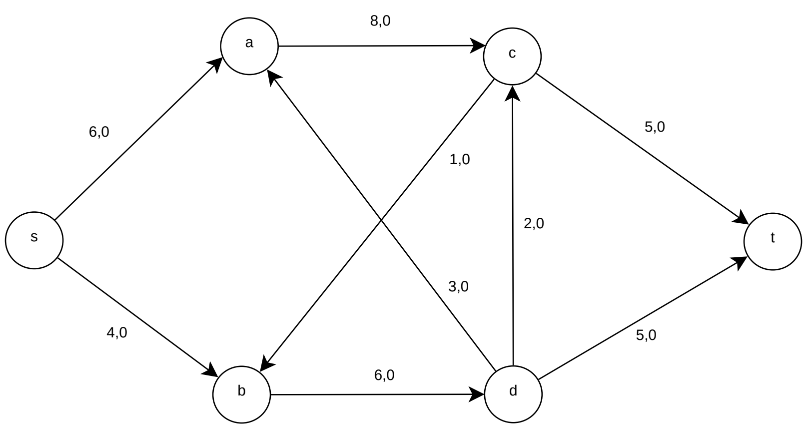


Demonstrate the main steps in the algorithm. Show all minimum cuts. How many different minimum cuts can you find?

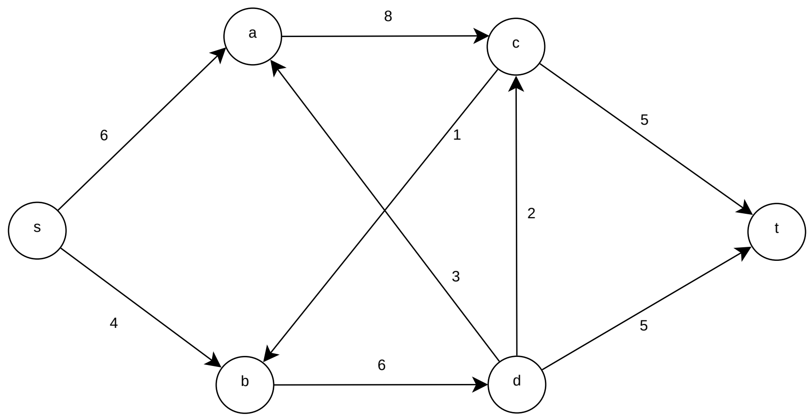
Solution:

Phase 1

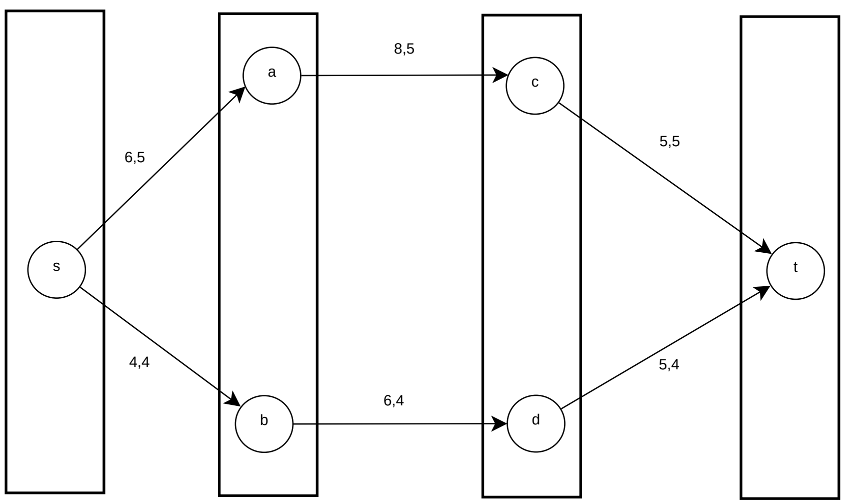
- Original network $\mathcal{N}$:



- Residual network $\mathcal{N}'$:

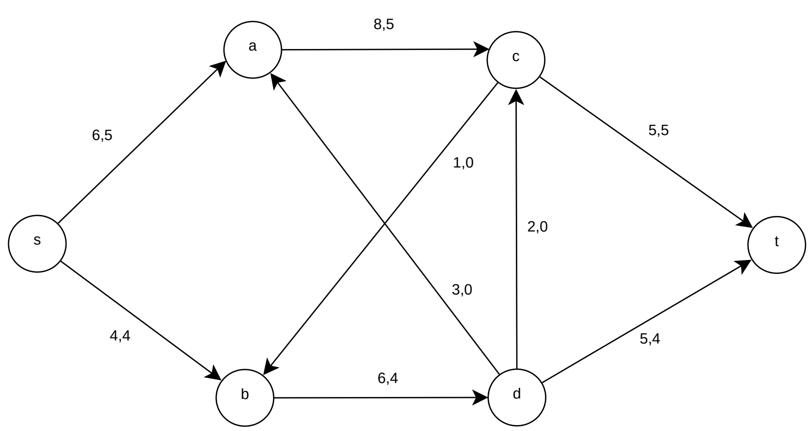


- Layered network $\mathcal{N}''$ with maximal flow:

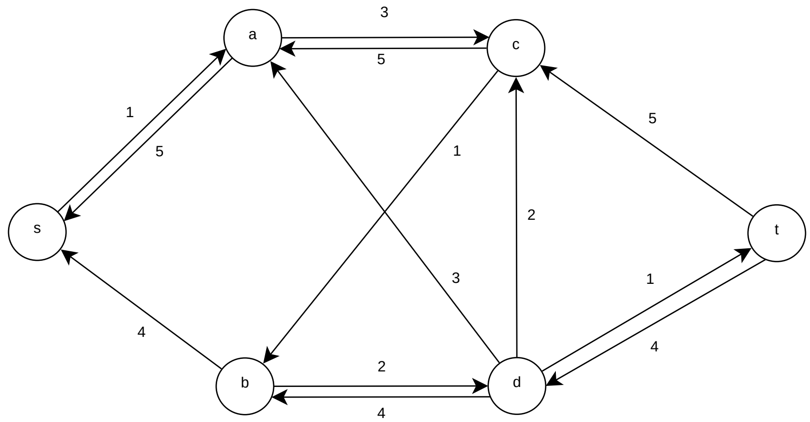


Phase 2

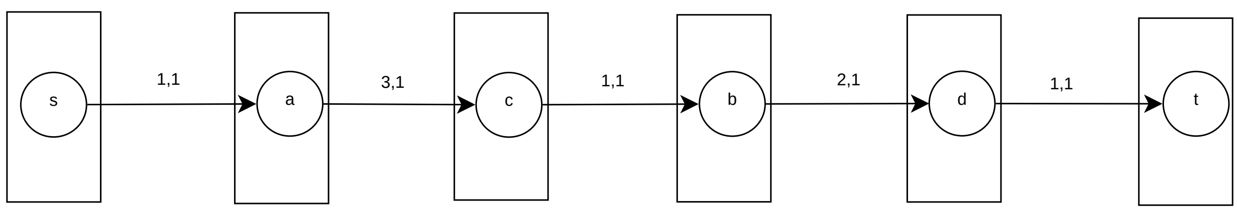
- Original network $\mathcal{N}$ with the updated flow:



- Residual network $\mathcal{N}'$:

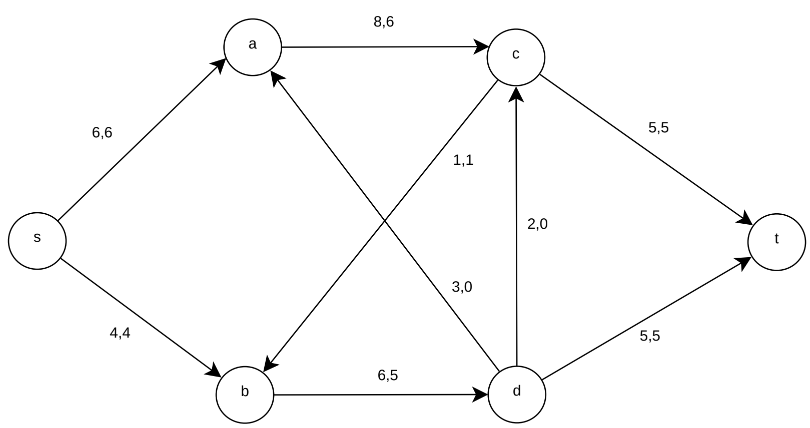


- Layered network $\mathcal{N}''$ and the corresponding maximal flow:

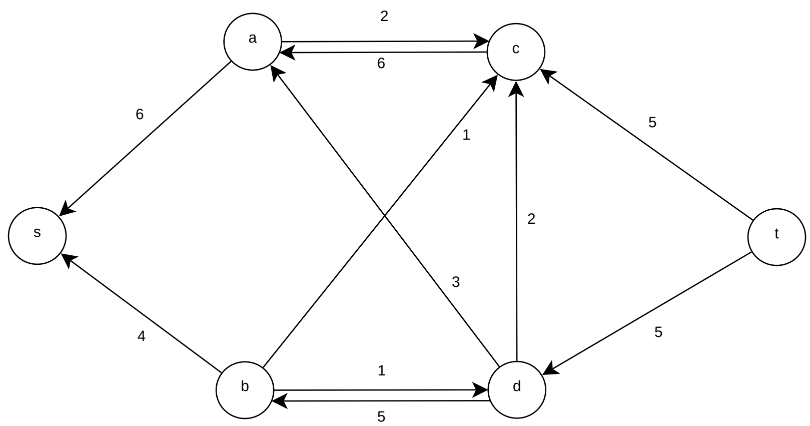


Phase 3

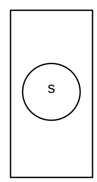
- Original network $\mathcal{N}$ and the updated flow:



- Residual network $\mathcal{N}'$:



- Layered network $\mathcal{N}''$:



We observe that t is not reachable anymore from s, and the layered network contains only one layer. Therefore, the algorithm halts. The maximum flow is as appears on the picture of network $\mathcal{N}$ in the beginning of Phase 3. The value of the maximum flow is 10.

Minimum cuts

The minimum cuts $(S : \overline{S})$ are obtained for $S = \{s\}$, $S = \{s, a, c\}$ and $S = \{s, a, b, c.d\}$. It can be seen that those are the only cuts that have capacity 10.

Question 2 (35 points). You are given the following linear-programming (LP) problem:

$$\begin{array}{lccccccc} \textbf{max} & 3x_1 & - & 2x_2 & - & 2x_3 & & \\ \textbf{s.t.} & x_1 & + & x_2 & & & \leq & 2 \\ & x_1 & & & - & x_3 & \leq & 2 \\ & & & x_2 & + & x_3 & \leq & 2 \\ & \multicolumn{7}{c}{x_1 \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0} \end{array}$$

(a) Solve this LP problem by using the simplex method.
(b) Formulate the dual LP problem.
(c) Verify that your solution of (a) is optimal by the substitution of the point $(2, 1, 0)$ into the dual problem. Explain your answer.

Solution:

(a) Write the given problem in the tableau form:

$$\left[\begin{array}{cccccc|c} \textcircled{1} & 1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & -1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & 2 \\ \hline 3 & -2 & -2 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

The only variable with the positive multiplier in the last row is x_1. Both entries (of the first column) in the first and second row have ratio 2, and thus we choose the encircled element as a pivot (it is also possible to chose as a pivot the element in the second row and first column instead).

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 2 \\ \hline 0 & -5 & -2 & -3 & 0 & 0 & -6 \end{array}\right] \begin{array}{l} (1) \leftarrow (1) \\ (2) \leftarrow (2) - (1) \\ (3) \leftarrow (3) \\ (4) \leftarrow (4) - 3 \cdot (1) \end{array}$$

Now, all coefficients in the last row are negative, and therefore the maximum of the objective function is 6. We have for this solution $x_1 = 2$, $x_2 = 0$ and $x_3 = 0$.

(b) By using the rules presented in the class, we write down the dual problem:

$$\begin{array}{lccccccc} \text{min} & 2y_1 & + & 2y_2 & + & 2y_3 & & \\ \text{s.t.} & y_1 & + & y_2 & & & \geq & 3 \\ & y_1 & & & + & y_3 & \geq & -2 \\ & & & -y_2 & + & y_3 & \geq & -2 \\ & \multicolumn{7}{c}{y_1 \geq 0, \; y_2 \geq 0, \; y_3 \geq 0} \end{array}$$

(c) Consider a substitution $y_1 = 2, y_2 = 1, y_3 = 0$. It is straightforward to see that this substitution satisfies all constraints of the dual problem, and therefore it is a feasible point. The value of the dual objective at this point is 6, which is equal to the value of the primal problem for $x_1 = 2, x_2 = 0, x_3 = 0$. By the strong duality theorem, indeed, 6 is the optimum value of both the primal and dual problem.

Question 3 (50 points). In this question, we consider the following variation of the bin packing problem. Assume that we are given n items with the real-valued sizes $a_1, a_2, \cdots, a_n \in (0, 1]$. Additionally, we are given a sufficiently large number of bins $b_1, b_2, \cdots$, where the bin b_i has capacity 2 if $i = 1, 3, 5, \cdots$, and b_i has capacity 1 if $i = 2, 4, 6, \cdots$. The goal is to find a packing of the items into the bins $b_1, b_2, \cdots, b_\ell$ (in that order), for the minimum possible ℓ.

We use the following First-Fit algorithm:

1. Order the items in the following order: $a_1, a_2, \cdots, a_n$.
2. For each $i = 1, 2, \cdots, n$, try to put the item a_i into the bin b_1, then into b_2, …, etc., until the item a_i fits in.
3. Output ℓ, the total number of bins used.

(a) Prove that at the end of the algorithm run, if $\ell > 2$, then the bins b_1 and b_2 together are full to the extent of at least 2 (i.e. the two bins together contain items whose total size is at least 2).

(b) Generalize (a) by observing that at the end of the algorithm run, if i is odd and $\ell > i+1$, then the bins b_i and b_{i+1} together are full to the extent of at least 2.

(c) Show that $\sum_{i=1}^{n} a_i \geq \ell - 1$ (hint: consider separately the cases when ℓ is odd and when ℓ is even).

(d) Prove that $\sum_{i=1}^{n} a_i \leq \frac{3}{2} \cdot \mathsf{OPT} + \frac{1}{2}$ (hint: consider separately odd and even values of OPT).

(e) Conclude that

$$\ell \leq \frac{3}{2} \cdot (\mathsf{OPT} + 1) .$$

Solution: Denote by s_i the total size of items in the bin b_i, for every i.

(a) Since $\ell > 2$, the bin b_2 is not empty. Let r be the size of the first item that is placed in b_2, and t be the total size of items in bin b_1 at the moment when the first item is placed in b_2.

The first item that was placed in b_2, was placed there because there was no sufficient place for it in b_1. Therefore, $t + r > 2$. After that, additional items could be added to the bins b_1 and b_2, and therefore $s_1 + s_2 \geq t + r > 2$.

(b) This question is very similar to (a). Since $\ell > i + 1$, the bin b_{i+1} is not empty. Let r be the size of the first item that is placed in b_{i+1}, and t be the total size of items in bin b_i at the moment when the first item is placed in b_{i+1}.

The first item that was placed in b_{i+1}, was placed there because there was no sufficient place for it in b_i. Therefore, $t + r > 2$. After that, additional items could be added to the bins b_i and b_{i+1}, and therefore $s_i + s_{i+1} \geq t + r > 2$.

(c) When all items are placed in ℓ bins, we have

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{\ell} s_j .$$

– If ℓ is odd, we have:

$$\sum_{j=1}^{\ell} s_j = (s_1+s_2)+(s_3+s_4)+\cdots+(s_{\ell-2}+s_{\ell-1})+s_\ell > \underbrace{2+2+\cdots+2}_{(\ell-1)/2}+s_\ell \geq (\ell-1)+s_\ell \geq \ell-1 ,$$

where inequality follows from part (b).

– If ℓ is even, we similarly have:

$$\sum_{j=1}^{\ell} s_j = (s_1+s_2)+(s_3+s_4)+\cdots+(s_{\ell-3}+s_{\ell-2})+(s_{\ell-1}+s_\ell)$$
$$> \underbrace{2+2+\cdots+2}_{(\ell-2)/2}+(s_{\ell-1}+s_\ell) \geq (\ell-2)+(s_{\ell-1}+s_\ell) .$$

Similarly to (a) and (b), one can observe that an item will be placed in the bin b_ℓ only if there if no sufficient place in bin $b_{\ell-1}$, and therefore $s_{\ell-1}+s_\ell > 2$. However, it is sufficient even to require a weaker condition that $s_{\ell-1}+s_\ell > 1$. To conclude, we obtain that in any case

$$\sum_{j=1}^{\ell} s_j > \ell - 1 .$$

(d) Consider an optimal assignment of items into bins, $\ell = \mathsf{OPT}$. We write:

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{\ell} s_j = \sum_{j \text{ is odd}} s_j + \sum_{j \text{ is even}} s_j .$$

Next, observe that for odd j, $s_j \leq 2$, and for even j, $s_j \leq 1$.

– If ℓ is odd, we have:

$$\sum_{j \text{ is odd}} s_j + \sum_{j \text{ is even}} s_j \leq \frac{\ell+1}{2}\cdot 2 + \frac{\ell-1}{2}\cdot 1 = \frac{3}{2}\ell + \frac{1}{2} .$$

– If ℓ is even, we have:

$$\sum_{j \text{ is odd}} s_j + \sum_{j \text{ is even}} s_j \leq \frac{\ell}{2}\cdot 2 + \frac{\ell}{2}\cdot 1 = \frac{3}{2}\ell < \frac{3}{2}\ell + \frac{1}{2} .$$

Since we consider the optimal placement of items, in both cases we obtain that

$$\sum_{i=1}^{n} a_i \leq \frac{3}{2}\cdot \mathsf{OPT} + \frac{1}{2} .$$

(e) By putting together the results of part (c) and part (d), we have

$$\ell - 1 \leq \sum_{i=1}^{n} a_i \leq \frac{3}{2} \cdot \mathsf{OPT} + \frac{1}{2} \ ,$$

which implies

$$\ell \leq \frac{3}{2} \cdot (\mathsf{OPT} + 1)$$

in a straightforward manner.