MTAT.03.286: Design and Analysis of Algorithms **University of Tartu**

Final exam

Course staff: Vitaly Skachek, Oliver-Matis Lill, Behzad Abdolmaleki, Eldho Thomas

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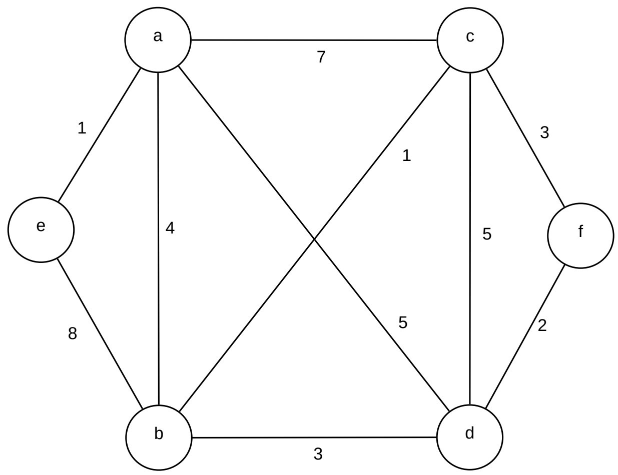
Student name: ____

Student ID: ____

1. This exam contains 10 pages. Check that no pages are missing.
2. It is possible to collect up to 110 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 1 hour and 40 minutes.
7. Good luck!

Question 1	
Question 2	
Question 3	
Total	

Question 1 (25 points). Find a minimum spanning tree (MST) in the following graph by using one of the algorithms that were studied in the course:



Show all the steps of the algorithm you are using.

Solution. Either Prim's or Kruskal's algorithm can be used to find the MST. This particular solution is by using Kruskal's algorithm.
Sorting the edges of the graph in non-decreasing order of their weights produces the following list:

$$(\{e,a\}, \{b,c\}, \{d,f\}, \{c,f\}, \{b,d\}, \{a,b\}, \{a,d\}, \{c,d\}, \{a,c\}, \{e,b\}) \ .$$

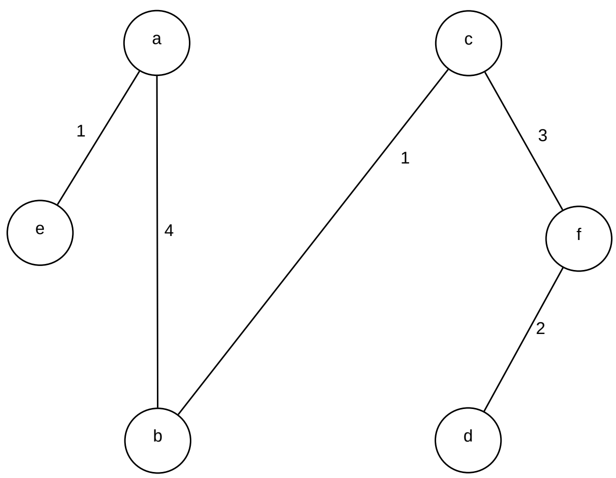
The corresponding weights are:

$$(1, 1, 2, 3, 3, 4, 5, 5, 7, 8) \ .$$

Execution of Steps 3-7 in the Kruskal's algorithm produces the following results:

i	Edge	Circuit-free?
1	{e,a}	Yes
2	{b,c}	Yes
3	{d,f}	Yes
4	{c,f}	Yes
5	{b,d}	No
6	{a,b}	Yes
7	{a,d}	No
8	{c,d}	No
9	{a,c}	No
10	{e,b}	No

The output is the following MST $\mathcal{T}$ with $w(\mathcal{T}) = 11$:


a
c
1
3
1
e
4
f
2
b
d

Question 2 (25 points).

By using Fast Fourier Transform (FFT) algorithm, evaluate the polynomial $P(x) = x^3+2x^2-3x-1$ at the complex 4th roots of unity. Show at least one level of recursion.

Solution. Let ω be the 4-th *p*rimitive root of unity. Then, for example, $\omega^4 = 1$, $\omega^2 = -1$, $\omega = i \triangleq \sqrt{-1}$ and $\omega^3 = -i$. As it was shown in the lecture, we can write the evaluation process in the matrix form:

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -3 \\ 2 \\ 1 \end{pmatrix}.$$

By permuting the order of the columns (odd-indexed and even-indexed columns are put together), we have

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^3 \\ \hline 1 & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^3 & \omega \end{array}\right) \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} & \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \end{array}\right) \cdot \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -i \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}\right) \cdot \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+(-2) \\ -3+(-4\cdot i) \\ \hline 1-(-2) \\ -3-(-4\cdot i) \end{pmatrix} = \begin{pmatrix} -1 \\ -3-4\cdot i \\ \hline 3 \\ -3+4\cdot i \end{pmatrix}.$$

Question 3 (60 points).

Definitions (reminder):

1. *A vertex cover* in an undirected finite graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a subset of vertices $\mathcal{C} \subseteq \mathcal{V}$, such that every edge in $\mathcal{E}$ has at least one of its vertices in $\mathcal{C}$.
2. Given an undirected finite graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, *a minimum vertex cover problem* is a problem of finding a vertex cover of minimum size (minimum number of vertices).
3. *A matching* in an undirected finite graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a subset of edges $\mathcal{M} \subseteq \mathcal{E}$, such that no two edges in $\mathcal{M}$ have joint vertices.
4. Given an undirected finite graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, *a maximum matching problem* is a problem of finding a matching of maximum size (maximum number of edges).
5. *An extreme point solution* is a feasible solution that cannot be expressed as a convex combination of other feasible solutions.

(a) Consider an undirected finite graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Prove that the size of any matching in $\mathcal{G}$ is smaller or equal to the size of any vertex cover in $\mathcal{G}$.

(b) Consider the following integer linear programming formulation for the minimum vertex cover problem:

$$\begin{aligned} \textsf{minimize} \quad & \sum_{v \in \mathcal{V}} x_v \\ \textsf{subject to} \quad & \forall e = \{u, v\} \in \mathcal{E} \;:\; x_u + x_v \geq 1 \\ & \forall v \in \mathcal{V} \;:\; x_v \in \{0, 1\} \end{aligned}$$

Here, for all $v \in \mathcal{V}$, we used indicator variables x_v, which are 1 if $v \in \mathcal{C}$, and 0 otherwise. Show a linear-programming relaxation of this problem, similarly to what was done in the class. Explain your answer.

(c) Formulate the maximum matching problem as an integer linear programming problem, and show that its LP relaxation is dual to the LP relaxation of the minimum vertex cover problem in (b).

(d) Reminder: we have shown in the homework assignment 6 that if the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is bipartite, then all extreme point solutions to the LP relaxation of the minimum vertex cover problem in (b) are integral (there is no need to show that again). It can also be shown that if the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is bipartite then all extreme point solutions to the LP relaxation of the maximum matching problem are integral.

By using those facts, prove that in the bipartite graph $\mathcal{G}$, the size of the maximum matching is *equal* to the size of the minimum vertex cover.

Solution.

(a) Let $\mathcal{M}$ be any matching in $\mathcal{G}$ and $\mathcal{C}$ be any vertex cover. Then, any edge $\{u, w\} \in \mathcal{M}$ must be covered by $\mathcal{C}$, and therefore either $u \in \mathcal{C}$ or $w \in \mathcal{C}$.

Denote the vertex that covers an edge $e \in \mathcal{E}$ by v_e, $v_e \in \mathcal{C}$ (if there are two of these vertices, pick one of them and denote it v_e).

Any two edges in $\mathcal{M}$ do not have common vertices by the definition of matching, and therefore every vertex covers at most one edge from $\mathcal{M}$. Consider all vertices v_e, $e \in \mathcal{E}$. They all in $\mathcal{C}$, and their number is equal to the size of $\mathcal{M}$. However, the cover may contain additional vertices. Therefore, the size of $\mathcal{C}$ is larger or equal to the size of $\mathcal{M}$. □

(b) The relaxation is:

$$\begin{aligned} \textsf{minimize} \quad & \sum_{v \in \mathcal{V}} x_v \\ \textsf{subject to} \quad & \forall e = \{u, v\} \in \mathcal{E} \ : \ x_u + x_v \geq 1 \\ & \forall v \in \mathcal{V} \ : \ 1 \geq x_v \geq 0 \end{aligned}$$

Here we replace integer indicator variables $x_v \in \{0, 1\}$ by continuous variables $x_v \in [0, 1]$.

Next, observe that the condition $x_v \leq 1$ can be omitted. Otherwise, if $x_v > 1$, its value can be decreased and the value of the objective function improved, without violating any constraints. We obtain:

$$\begin{aligned} \textsf{minimize} \quad & \sum_{v \in \mathcal{V}} x_v \\ \textsf{subject to} \quad & \forall e = \{u, v\} \in \mathcal{E} \ : \ x_u + x_v \geq 1 \\ & \forall v \in \mathcal{V} \ : \ x_v \geq 0 \end{aligned}$$

(c) Define indicator variables y_e for all edges $e \in \mathcal{E}$, where $y_e = 1$ if $e \in \mathcal{M}$, and $y_e = 0$ if $e \notin \mathcal{M}$. By the definition of the matching, two edges in $\mathcal{M}$ cannot have a joint vertex. Therefore, for each vertex $v \in \mathcal{V}$, we obtain a constraint of the form

$$\sum_{e \,:\, v \in e} y_e \leq 1 \ .$$

The objective is to maximize the number of edges in the matching. We obtain the following integer LP problem:

$$\begin{aligned} \textsf{maximize} \quad & \sum_{e \in \mathcal{E}} y_e \\ \textsf{subject to} \quad & \forall v \in \mathcal{V} \ : \ \sum_{e \,:\, v \in e} y_e \leq 1 \\ & \forall e \in \mathcal{E} \ : \ y_e \in \{0, 1\} \end{aligned}$$

We relax the condition $y_e \in \{0, 1\}$ by turning it into $0 \leq y_e \leq 1$. Now, we observe that if $y_e > 1$, then any constraint containing y_e is violated. Therefore, it suffices to require that

$y_e \geq 0$ instead. To conclude, we have:

$$\begin{aligned} \textsf{maximize} \quad & \sum_{e \in \mathcal{E}} y_e \\ \textsf{subject to} \quad & \forall v \in \mathcal{V} \ : \ \sum_{e \,:\, v \in e} y_e \leq 1 \\ & \forall e \in \mathcal{E} \ : \ y_e \geq 0 \end{aligned}$$

It can be verified that this problem is dual to the LP problem describing the minimum cover problem. The minimization is replaced by maximization, the variables correspond to inequalities and vice versa, and the matrix describing relations between equations and variables has been transposed. Therefore, these two LP problems are dual.

(d) Assume that the graph $\mathcal{G}$ is bipartite. As it is shown in homework assignment 6, all extreme point solutions of the LP problem for the vertex cover are integral. Denote by $\mathsf{OPT}_f(\mathcal{C})$ the minimum fractional solution to the minimum vertex cover LP problem, and by $\mathsf{OPT}(\mathcal{C})$ the minimum size integral solution to the minimum vertex cover ILP problem. From homework assignment 6 we have that $\mathsf{OPT}_f(\mathcal{C})$ =$\mathsf{OPT}(\mathcal{C})$.

Similarly, denote by $\mathsf{OPT}_f(\mathcal{M})$ the maximum fractional solution to the maximum matching LP problem, and by $\mathsf{OPT}(\mathcal{M})$ the maximum size integral solution to the maximum vertex cover ILP problem. As it is mentioned in the guidance to the question, the extreme point solutions are integral, and therefore $\mathsf{OPT}_f(\mathcal{M})$ =$\mathsf{OPT}(\mathcal{M})$.

By the strong duality theorem, and by duality proved in part (c), we have that

$$\mathsf{OPT}_f(\mathcal{M}) = \mathsf{OPT}_f(\mathcal{C}) \ .$$

Let $\mathcal{M}$ be a maximum matching in $\mathcal{G}$ and $\mathcal{C}$ be a minimum vertex cover in $\mathcal{G}$. Then, $|\mathcal{M}| = \mathsf{OPT}(\mathcal{M})$ and $\mathsf{OPT}(\mathcal{C}) = |\mathcal{C}|$. By putting all these together, we obtain that:

$$|\mathcal{M}| = \mathsf{OPT}(\mathcal{M}) = \mathsf{OPT}_f(\mathcal{M}) = \mathsf{OPT}_f(\mathcal{C}) = \mathsf{OPT}(\mathcal{C}) = |\mathcal{C}| \ .$$

□