MTAT.03.286: Design and Analysis of Algorithms **University of Tartu**

Final exam

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Student name: ____

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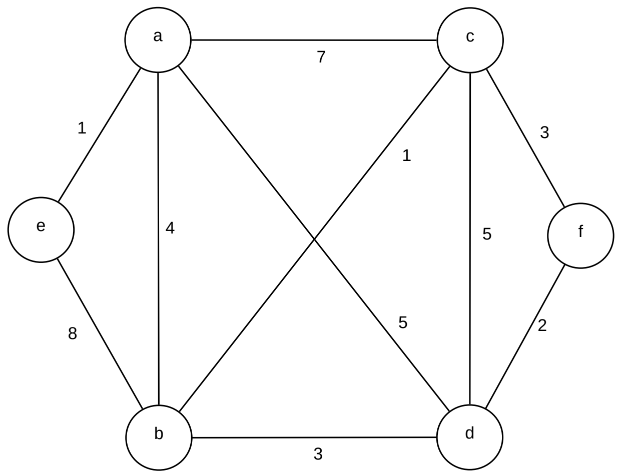
1. This exam contains 10 pages. Check that no pages are missing.
2. It is possible to collect up to 110 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 1 hour and 40 minutes.
7. Good luck!

Question 1	
Question 2	
Question 3	
Total	

Question 1 (25 points). Find a minimum spanning tree (MST) in the following graph by using one of the algorithms that were studied in the course:



Show all the steps of the algorithm you are using.

Question 2 (25 points).

By using Fast Fourier Transform (FFT) algorithm, evaluate the polynomial $P(x) = x^3 + 2x^2 - 3x - 1$ at the complex 4th roots of unity. Show at least one level of recursion.

Question 3 (60 points).

Definitions (reminder):

1. *A vertex cover* in an undirected finite graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is a subset of vertices $\mathcal{C} \subseteq \mathcal{V}$, such that every edge in $\mathcal{E}$ has at least one of its vertices in $\mathcal{C}$.
2. Given an undirected finite graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, *a minimum vertex cover problem* is a problem of finding a vertex cover of minimum size (minimum number of vertices).
3. *A matching* in an undirected finite graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is a subset of edges $\mathcal{M} \subseteq \mathcal{E}$, such that no two edges in $\mathcal{M}$ have joint vertices.
4. Given an undirected finite graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, *a maximum matching problem* is a problem of finding a matching of maximum size (maximum number of edges).
5. *An extreme point solution* is a feasible solution that cannot be expressed as a convex combination of other feasible solutions.

(a) Consider an undirected finite graph $\mathcal{G}(\mathcal{V},\mathcal{E})$. Prove that the size of any matching in $\mathcal{G}$ is smaller or equal to the size of any vertex cover in $\mathcal{G}$.

(b) Consider the following integer linear programming formulation for the minimum vertex cover problem:

$$\begin{aligned} \textsf{minimize} \quad & \sum_{v\in\mathcal{V}} x_v \\ \textsf{subject to} \quad & \forall e = \{u,v\} \in \mathcal{E} \;:\; x_u + x_v \geq 1 \\ & \forall v \in \mathcal{V} \;:\; x_v \in \{0,1\} \end{aligned}$$

Here, for all $v \in \mathcal{V}$, we used indicator variables x_v, which are 1 if $v \in \mathcal{C}$, and 0 otherwise. Show a linear-programming relaxation of this problem, similarly to what was done in the class. Explain your answer.

(c) Formulate the maximum matching problem as an integer linear programming problem, and show that its LP relaxation is dual to the LP relaxation of the minimum vertex cover problem in (b).

(d) Reminder: we have shown in the homework assignment 6 that if the graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is bipartite, then all extreme point solutions to the LP relaxation of the minimum vertex cover problem in (b) are integral (there is no need to show that again). It can also be shown that if the graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is bipartite then all extreme point solutions to the LP relaxation of the maximum matching problem are integral.

By using those facts, prove that in the bipartite graph $\mathcal{G}$, the size of the maximum matching is *equal* to the size of the minimum vertex cover.