MTAT.03.286: Design and Analysis of Algorithms **University of Tartu**

Final exam

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Student name: ____

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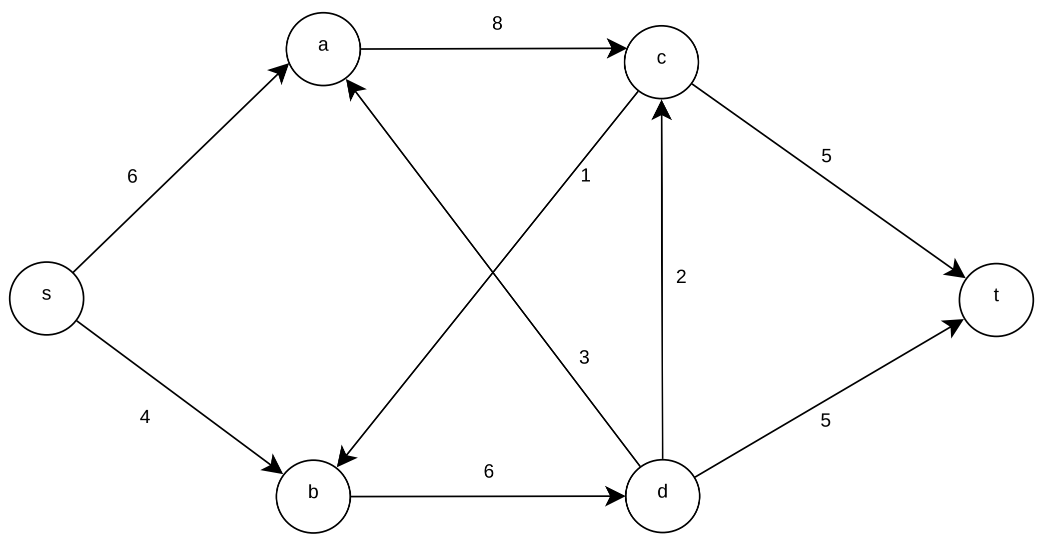
1. This exam contains 8 pages. Check that no pages are missing.
2. It is possible to collect up to 110 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 1 hour and 40 minutes.
7. Good luck!

Question 1	
Question 2	
Question 3	
Total	

Question 1 (25 points). Find a maximum flow between s and t in the following network by using Dinitz algorithm:



Demonstrate the main steps in the algorithm. Show all minimum cuts. How many different minimum cuts can you find?

Question 2 (35 points). You are given the following linear-programming (LP) problem:

$$\begin{array}{lrcrcrcl}
\textbf{max} & 3x_1 & - & 2x_2 & - & 2x_3 & & \\
\textbf{s.t.} & x_1 & + & x_2 & & & \leq & 2 \\
 & x_1 & & & - & x_3 & \leq & 2 \\
 & & & x_2 & + & x_3 & \leq & 2 \\
 & \multicolumn{7}{c}{x_1 \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0}
\end{array}$$

(a) Solve this LP problem by using the simplex method.

(b) Formulate the dual LP problem.

(c) Verify that your solution of (a) is optimal by the substitution of the point $(2, 1, 0)$ into the dual problem. Explain your answer.

Question 3 (50 points). In this question, we consider the following variation of the bin packing problem. Assume that we are given n items with the real-valued sizes $a_1, a_2, \cdots, a_n \in (0, 1]$. Additionally, we are given a sufficiently large number of bins $b_1, b_2, \cdots$, where the bin b_i has capacity 2 if $i = 1, 3, 5, \cdots$, and b_i has capacity 1 if $i = 2, 4, 6, \cdots$. The goal is to find a packing of the items into the bins $b_1, b_2, \cdots, b_\ell$ (in that order), for the minimum possible ℓ.

We use the following First-Fit algorithm:

1. Order the items in the following order: $a_1, a_2, \cdots, a_n$.
2. For each $i = 1, 2, \cdots, n$, try to put the item a_i into the bin b_1, then into b_2, ..., etc., until the item a_i fits in.
3. Output ℓ, the total number of bins used.

(a) Prove that at the end of the algorithm run, if $\ell > 2$, then the bins b_1 and b_2 together are full to the extent of at least 2 (i.e. the two bins together contain items whose total size is at least 2).

(b) Generalize (a) by observing that at the end of the algorithm run, if i is odd and $\ell > i+1$, then the bins b_i and b_{i+1} together are full to the extent of at least 2.

(c) Show that $\sum_{i=1}^{n} a_i \geq \ell - 1$ (hint: consider separately the cases when ℓ is odd and when ℓ is even).

(d) Prove that $\sum_{i=1}^{n} a_i \leq \frac{3}{2} \cdot \mathsf{OPT} + \frac{1}{2}$ (hint: consider separately odd and even values of OPT).

(e) Conclude that

$$\ell \leq \frac{3}{2} \cdot (\mathsf{OPT} + 1) \ .$$