MTAT.03.286: Advanced Methods in Algorithms University of Tartu

Some Example Algorithms

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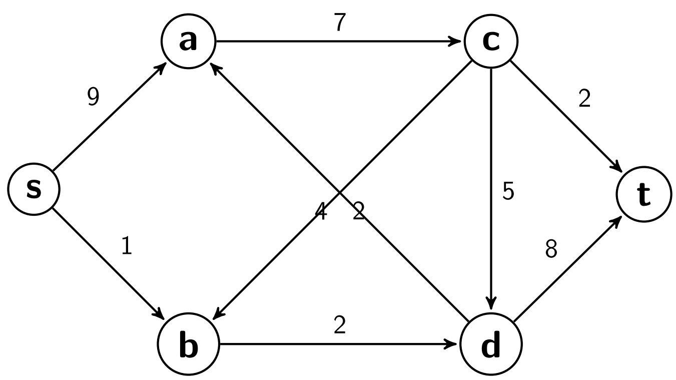
1 Dinitz algorithm

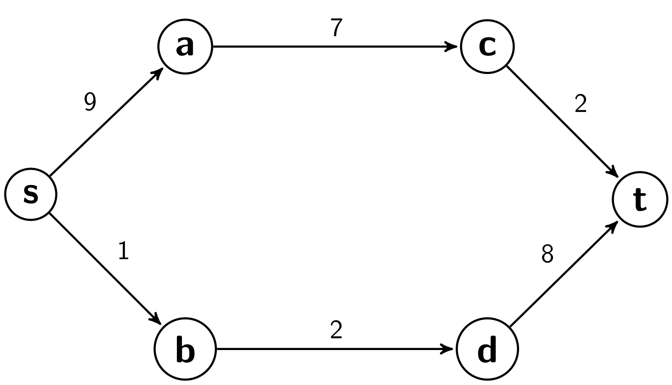
Find a maximum flow between s and t in the following network by using Dinitz algorithm:



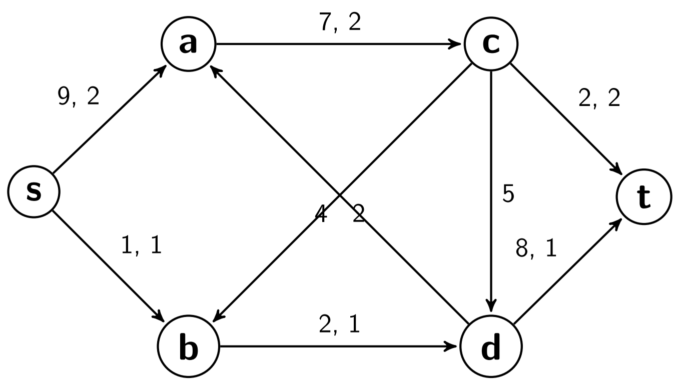
Demonstrate the main steps in the algorithm. Show all minimum cuts. How many different minimum cuts can you find?

1.1 Dinitz algorithm

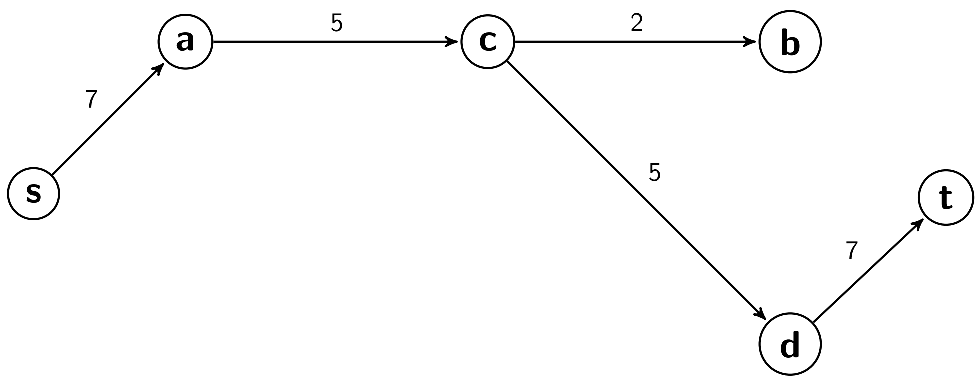
The first layered network is as follows



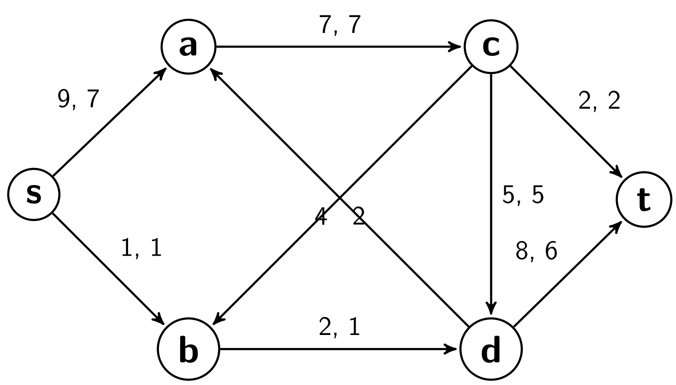
Here we can push flow 2 on the path $s-a-c-t$ and flow 1 on $s-b-d-t$, giving us the following graph



This gives the following layered network



That allows to push flow of size 5 on the path $s - a - c - d - t$ to give the following graph



This flow is maximum and has value $F = 8$, if building another layered network then the will be no path to t from s.

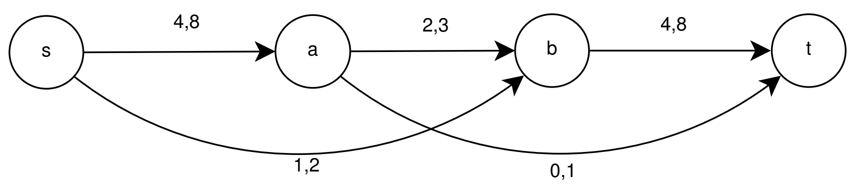
1.2 Minumum cuts

The fact that maximum flow has size $F = 8$ is illustrated by minimum cuts of size 8. Such a cut is defined by $S = \{s, a\}$.

This is the only such cut.

2 Flow with upper and lower limits

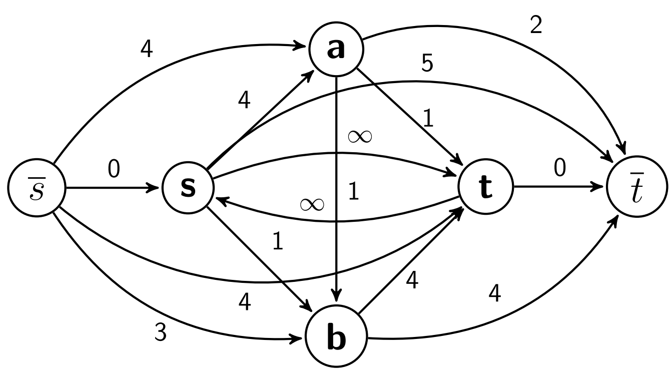
Find a legal flow from s to t in the following network with upper and lower bounds. (You don't have to specify all the steps in Ford-Flukerson or Dinitz algorithm that you are using, but you have to explain the construction and the resulting flow.)



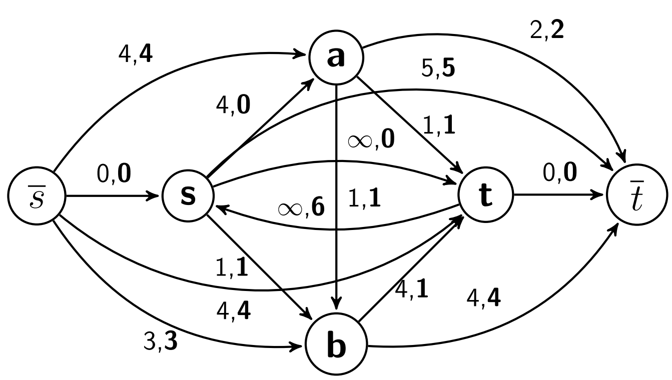
Find a maximum flow in the network in part (a). Show all minimum cuts.

2.1 Legal flow

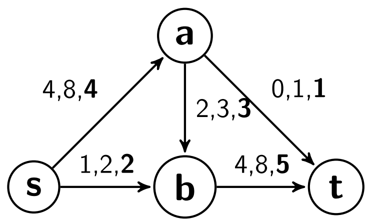
To find a legal flow we add a new sink and source and modify the capacities of the existing edges. This gives the graph.



We can find a maximal flow in this new graph using any maximum flow algorithms. The outcome is as follows, where the maximality can be observed for example using the cut ...



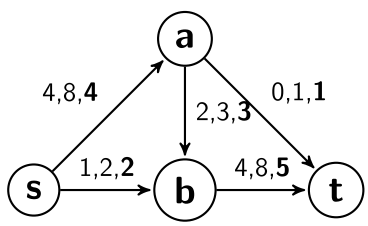
According to the algorithm, this translates to the following legal flow in the original graph.



2.2 Maximal flow

From the previous legal flow we can maximize this using a modification of the Ford-Fulkerson algorithm. When finding an augmenting path, the edge can be traversed backwards only as long as the flow on the edge remains valid with respect to the lower bound $f(e) \geq b(e)$. The forward edge can be used as long as it satisfies the capacity.

This gives the following maximum flow as there actually are no more augmenting paths.



The value of the flow is 6 and its maximality is indicated also by minumum cut $S = \{s, a\}$, which is the only minimum cut for this graph.

3 Polynomial evaluation

Evaluate the polynomial $A(x) = 3x^4 + 2x^3 - 2x^2 - x$ at the 6'th roots of unity.

Let $\omega^6 = 1$ where we choose elements using the polar coordinates as $\omega = (1, \frac{2\pi}{6})$, $\omega^2 = (1, \frac{4\pi}{6})$, $\omega^3 = (1, \pi) = -1$, $\omega^4 = (\frac{8\pi}{6})$, $\omega^5 = (1, \frac{10\pi}{6})$, $\omega^6 = (1, 2\pi) = 1$. By $M_n(x)$ we denote the $n \times n$ Vandermonde matrix of variable x.

We use the matrix notation to evaluate the polynomial $A(x)$ at the previously marked points.

We do FFT and reorder the columns in the matrix on the second line.

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \\ A(\omega^4) \\ A(\omega^5) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^1 & \omega^4 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ -2 \\ 2 \\ 3 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 & \omega^5 \\ 1 & \omega^4 & \omega^2 & \omega^2 & 1 & \omega^4 \\ 1 & 1 & 1 & \omega^3 & \omega^3 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^4 & 1 & \omega^2 \\ 1 & \omega^4 & \omega^2 & \omega^5 & \omega^3 & \omega \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \\ 3 \\ -1 \\ 2 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} M_3(\omega^2) & E \cdot M_3(\omega^2) \\ M_3(\omega^2) & -F \cdot M_3(\omega^2) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \\ 3 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

Here

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad F = \begin{pmatrix} \omega^3 & 0 & 0 \\ 0 & \omega^4 & 0 \\ 0 & 0 & \omega^5 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & -\omega^2 \end{pmatrix} = -E$$

This helps us to reduce the initial problem to computing

$$M_3(\omega^2) \cdot \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \qquad M_3(\omega^2) \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

And additional multiplications with E and $-E$.

Now we just evaluate the necessary multiplications

$$M_3(\omega^2) \cdot \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2\omega^2 - 3\omega \\ 2\omega + 3\omega^2 \end{pmatrix} \qquad M_3(\omega^2) \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 + 2\omega^2 \\ -1 - 2\omega \end{pmatrix}$$

$$E \cdot \begin{pmatrix} 1 \\ -1 + 2\omega^2 \\ -1 - 2\omega \end{pmatrix} = \begin{pmatrix} 1 \\ -1\omega - 2 \\ -\omega^2 + 2 \end{pmatrix} \qquad -E \cdot \begin{pmatrix} 1 \\ -1 + 2\omega^2 \\ -1 - 2\omega \end{pmatrix} = \begin{pmatrix} -1 \\ \omega + 2 \\ \omega^2 - 2 \end{pmatrix}$$

In total this gives us

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \\ A(\omega^4) \\ A(\omega^5) \end{pmatrix} = \begin{pmatrix} 1+1 \\ -2\omega^2 - 3\omega - \omega - 2 \\ 2\omega + 3\omega^2 - \omega^2 + 2 \\ 1-1 \\ -2\omega^2 - 3\omega + \omega + 2 \\ 2\omega + 3\omega^2 + \omega^2 - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2\omega^2 - 4\omega - 2 \\ 2\omega^2 + 2\omega + 2 \\ 0 \\ -2\omega^2 - 2\omega^2 + 2 \\ 4\omega^2 + 2\omega - 2 \end{pmatrix}$$

4 Polynomial evaluation

By using the Fast Fourier Transform (FFT) algorithm, evaluate the polynomial $A(x) = x^5 + 2x^2 - x - 1$ *at the complex* 8*th roots of unity. Show at least one level of recursion.*

Let $\omega = (1, \frac{2\pi}{8})$ be one root, that generates all the roots of unity. We have $\omega^4 = -1$ and $\omega^2 = i$. We can do the evaluation in the algebraic mode by creating the Vandermonde matrix that we denote by $M_8(\omega)$ and multiplying it with the vector encoding of the polynomial. Then we do the separation of even and odd columns in the matrix.

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \\ A(\omega^4) \\ A(\omega^5) \\ A(\omega^6) \\ A(\omega^7) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 & \omega^7 \\ 1 & \omega^4 & 1 & \omega^4 & \omega^2 & \omega^6 & \omega^2 & \omega^6 \\ 1 & \omega^6 & \omega^4 & \omega^2 & \omega^3 & \omega & \omega^7 & \omega^5 \\ 1 & 1 & 1 & 1 & \omega^4 & \omega^4 & \omega^4 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^5 & \omega^7 & \omega & \omega^3 \\ 1 & \omega^4 & 1 & \omega^4 & \omega^6 & \omega^2 & \omega^6 & \omega^2 \\ 1 & \omega^6 & \omega^4 & \omega^2 & \omega^7 & \omega^5 & \omega^3 & \omega \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

At this stage we can see the recursive step occurring. Especially, the second stage has the following format

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \\ A(\omega^4) \\ A(\omega^5) \\ A(\omega^6) \\ A(\omega^7) \end{pmatrix} = \begin{pmatrix} M_4(\omega^2) & A \cdot M_4(\omega^2) \\ M_4(\omega^2) & -A \cdot M_4(\omega^2) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

where

$$M_4(\omega^2) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^6 & \omega^4 & \omega^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{pmatrix}$$

Hence, recursively we need to compute

$$M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

and

$$A \cdot M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

as

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \\ A(\omega^4) \\ A(\omega^5) \\ A(\omega^6) \\ A(\omega^7) \end{pmatrix} = \begin{pmatrix} M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} & + & A \cdot M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} & - & A \cdot M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

.

We could do these computations with further recursive steps, however, for now we just do the required multiplications using traditional matrix multiplication.

$$M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+2i \\ -3 \\ -1-2i \end{pmatrix}$$

$$\begin{aligned} A \cdot M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} &= A \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = A \cdot \begin{pmatrix} 0 \\ -2 \\ 0 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2\omega \\ 0 \\ -2\omega^3 \end{pmatrix} \end{aligned}$$

Finally, we but this back together to obtain the required evaluation.

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \\ A(\omega^4) \\ A(\omega^5) \\ A(\omega^6) \\ A(\omega^7) \end{pmatrix} = \begin{pmatrix} M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} & + & A \cdot M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} & - & A \cdot M_4(\omega^2) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 \\ 1+2i \\ -3 \\ -1-2i \end{pmatrix} & + & \begin{pmatrix} 0 \\ -2\omega \\ 0 \\ -2\omega^3 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1+2i \\ -3 \\ -1-2i \end{pmatrix} & - & \begin{pmatrix} 0 \\ -2\omega \\ 0 \\ -2\omega^3 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1-2\omega+2\omega^2 \\ -3 \\ -1-2\omega^2-2\omega^3 \\ -1 \\ -1+2\omega+2\omega^2 \\ 3 \\ -1-2\omega^2+2\omega^3 \end{pmatrix}$$

5 Polynomial multiplication

Let $A(x) = x^2+x+2$ and $B(x) = 2x+3$. Evaluate the polynomial multiplication $C(x) = A(x) \cdot B(x)$.

Number of points

We need at least 4 points, because the degree of $C(x)$ will be 3. Hence, it has four coefficients and is described by four points.

Evaluate $A(X)$

We choose the 4th roots of unity so that $\omega^4 = 1$ and $\omega = i$. Hence $\omega^2 = -1$, $\omega^3 = -i$.

At first we write out the FFT matrix and then we rearrange the columns.

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^3 \\ 1 & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^3 & \omega \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} M_2(\omega^2) & E \cdot M_2(\omega^2) \\ M_2(\omega^2) & F \cdot M_2(\omega^2) \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Here

$$E = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \qquad F = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix} == \begin{pmatrix} -1 & 0 \\ 0 & -\omega \end{pmatrix} = -E$$

We should now recursively evaluate

$$M_2(\omega^2) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad M_2(\omega^2) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We can write this out using FFT to reduce this to multiplying with $M_1 = 1$.

$$M_2(\omega^2) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$M_2(\omega^2) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now, we can put this together

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 3+1 \\ 1+\omega \\ 3-1 \\ 1-\omega \end{pmatrix} = \begin{pmatrix} 4 \\ 1+i \\ 2 \\ 1-i \end{pmatrix}$$

Evaluate $B(x)$

This is analogous to the evaluation of $A(x)$

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^3 \\ 1 & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^3 & \omega \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} =$$
$$= \begin{pmatrix} M_2(\omega^2) & E \cdot M_2(\omega^2) \\ M_2(\omega^2) & F \cdot M_2(\omega^2) \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$M_2(\omega^2) \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$M_2(\omega^2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Which in total gives

$$\begin{pmatrix} B(1) \\ B(\omega) \\ B(\omega^2) \\ B(\omega^3) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 3+2 \\ 3+2\omega \\ 3-2 \\ 3-2\omega \end{pmatrix} = \begin{pmatrix} 5 \\ 3+2i \\ 1 \\ 3-2i \end{pmatrix}$$

Compute $C(x)$ at 4th roots of unity

By definition we know that $C(x_i) = A(x_i) \cdot B(x_i)$ for any x_i. Therefore

$$\begin{aligned} C(1) &= A(1) \cdot B(1) = 4 \cdot 5 = 20 \\ C(\omega) &= A(\omega) \cdot B(\omega) = (1+i)(3+2i) = 1+5i \\ C(\omega^2) &= A(\omega^2) \cdot B(\omega^2) = 2 \cdot 1 = 2 \\ C(\omega^3) &= A(\omega^3) \cdot B(\omega^3) = (1-i)(3-2i) = 1-5i \end{aligned}$$

Find the coefficients of $C(x)$

We need to find c_0, c_1, c_2, c_3 such that $C(x) = c_3x^3 + c_2x^2 + c_1x + c_0$. To find the coefficients we need to find the inverse of $M_4(\omega)$. It is a Vandermonde matrix, hence,

$$M_4(\omega)^{-1} = \frac{1}{4} \cdot M_4(-\omega)$$

As we know that

$$\begin{pmatrix} 20 \\ 1+5i \\ 2 \\ 1-5i \end{pmatrix} = M_4(\omega) \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Therefore, we could compute this using the FFT process, but for now we just compute it using common matrix multiplication.

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = M_4(\omega)^{-1} \cdot \begin{pmatrix} 20 \\ 1+5i \\ 2 \\ 1-5i \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^3 & \omega^2 & \omega \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega & \omega^2 & \omega^3 \end{pmatrix} \cdot \begin{pmatrix} 20 \\ 1+5i \\ 2 \\ 1-5i \end{pmatrix} =$$

$$= \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 20 \\ 1+5i \\ 2 \\ 1-5i \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 5 \\ 2 \end{pmatrix}$$

Therefore, we have $C(x) = 2x^3 + 5x^2 + 7x + 6$.

6 Polynomial multiplication

Let $A(x) = x-1$ and $B(x) = x^2-x+2$. Evaluate the polynomial multiplication $C(x) = A(x) \cdot B(x)$.

Number of points

We need at least 4 points, because the degree of $C(x)$ will be 3. Hence, it has four coefficients and is described by four points.

Evaluate $A(x)$

We choose the 4th roots of unity so that $\omega^4 = 1$ and $\omega = i$. Hence $\omega^2 = -1$, $\omega^3 = -i$.

At first we write out the FFT matrix and then we rearrange the columns.

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^3 \\ 1 & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^3 & \omega \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} M_2(\omega^2) & E \cdot M_2(\omega^2) \\ M_2(\omega^2) & F \cdot M_2(\omega^2) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Here

$$E = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \qquad F = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix} == \begin{pmatrix} -1 & 0 \\ 0 & -\omega \end{pmatrix} = -E$$

We should now recursively evaluate

$$M_2(\omega^2) \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad M_2(\omega^2) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We can write this out using FFT to reduce this to multiplying with $M_1 = 1$.

$$M_2(\omega^2) \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$M_2(\omega^2) \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now, we can put this together

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -1+1 \\ -1+\omega \\ -1-1 \\ -1-\omega \end{pmatrix} = \begin{pmatrix} 0 \\ -1+i \\ -2 \\ -1-i \end{pmatrix}$$

Evaluate $B(x)$

This is analogous to the evaluation of $A(x)$

$$\begin{pmatrix} A(1) \\ A(\omega) \\ A(\omega^2) \\ A(\omega^3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^3 \\ 1 & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^3 & \omega \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} M_2(\omega^2) & E \cdot M_2(\omega^2) \\ M_2(\omega^2) & F \cdot M_2(\omega^2) \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$M_2(\omega^2) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$M_2(\omega^2) \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & \omega^2 M_1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Which in total gives

$$\begin{pmatrix} B(1) \\ B(\omega) \\ B(\omega^2) \\ B(\omega^3) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 3-1 \\ 1-\omega \\ 3+1 \\ 1+\omega \end{pmatrix} = \begin{pmatrix} 2 \\ 1-i \\ 4 \\ 1+i \end{pmatrix}$$

Compute $C(x)$ **at 4th roots of unity**

By definition we know that $C(x_i) = A(x_i) \cdot B(x_i)$ for any x_i. Therefore

$$\begin{aligned} C(1) &= A(1) \cdot B(1) = 0 \cdot 2 = 0 \\ C(\omega) &= A(\omega) \cdot B(\omega) = (-1+i)(1-i) = 2i \\ C(\omega^2) &= A(\omega^2) \cdot B(\omega^2) = -2 \cdot 4 = -8 \\ C(\omega^3) &= A(\omega^3) \cdot B(\omega^3) = (-1-i)(1+i) = -2i \end{aligned}$$

Find the coefficients of $C(x)$

We need to find c_0, c_1, c_2, c_3 such that $C(x) = c_3x^3 + c_2x^2 + c_1x + c_0$. To find the coefficients we need to find the inverse of $M_4(\omega)$. It is a Vandermonde matrix, hence,

$$M_4(\omega)^{-1} = \frac{1}{4} \cdot M_4(-\omega)$$

As we know that

$$\begin{pmatrix} 0 \\ 2i \\ -8 \\ -2i \end{pmatrix} = M_4(\omega) \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Therefore, we could compute this using the FFT process, but for now we just compute it using common matrix multiplication.

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = M_4(\omega)^{-1} \cdot \begin{pmatrix} 20 \\ 1+5i \\ 2 \\ 1-5i \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^3 & \omega^2 & \omega \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega & \omega^2 & \omega^3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2i \\ -8 \\ -2i \end{pmatrix} =$$

$$= \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2i \\ -8 \\ -2i \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} -8 \\ 12 \\ -8 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -2 \\ 1 \end{pmatrix}$$

Therefore, we have $C(x) = x^3 - 2x^2 + 3x - 2$.

7 Simplex algorithm

Solve the following linear-programming problem using simplex algorithm:

$$\begin{array}{rl} \textbf{max} & x_1 + 7x_2 + x_3 \\ \textbf{s.t.} & x_1 + x_2 \leq 3 \\ & 2x_2 + x_3 \leq 4 \\ & x_1 + x_3 \leq 2 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

7.1 Tableau format

We write down the initial problem in the tableau format and then use this to perform the simplex algorithm on this format. The first part is the matrix A representing the left hand side of the constraints. The middle part stands for the slack variables and the last column corresponds to the right hand side of the constraints $\vec{b}$. The last row holds the objective function information, especially it begins with $\vec{c}$.

$$\left(\begin{array}{ccc|ccc|c} 1 & 1 & 0 & 1 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 & 1 & 0 & 4 \\ 1 & 0 & 1 & 0 & 0 & 1 & 2 \\ \hline 1 & 7 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

7.2 Steps of the simplex algorithm

The simplex algorithm in requires us to follow two rules and do Gaussian elimination.

1. Choose a column (variable) with a positive coefficient in the last row. If no such column exists then the current solution is optimal.
2. For each row i where this column c has a positive entry, compute the ratio $\frac{b[i]}{c[i]}$ between this entry and the right hand side of the corresponding constraint. Choose the row with the smallest ratio.

The chosen element is used as a pivot and then Gaussian elimination is performed.

First we can choose column 1, 2 or 3. Let's choose 2 and the respective ratios are $\frac{3}{2}$ and $\frac{4}{2}$ meaning that we have to select row 2. We obtain the following matrix after applying the Gaussian elimination. (First row is (1)-$\frac{1}{2}$(2), Second row is $\frac{1}{2}$(2), Third remains the same and Fourth row is (4)-$\frac{7}{2}$(2)).

$$\left(\begin{array}{ccc|ccc|c} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 & 1 & 2 \\ \hline 1 & 0 & -\frac{5}{2} & 0 & -\frac{7}{2} & 0 & -14 \end{array}\right)$$

Now the only column that we can choose is the first column and based on the respective ratios, we have to take the first row. The second row remains the same, but we need to subtract the first row from the thirds and fourth row.

$$\left(\begin{array}{ccc|ccc|c} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 2 \\ 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 1 \\ \hline 0 & 0 & -2 & -1 & -3 & 0 & -15 \end{array}\right)$$

In here we can not do any more steps as all variables have negative entries in the last row.

7.3 Result

The result of the Simplex algorithm can be written out from the final tableau based on the basic variables. Right now the basic variables are x_1, x_2 and z_3 as these are the variables with a single 1 in the respective columns. We can find the values for these variables in the last column of in the row where the respective columns have 1. Hence, the optimum is obtained at $x_1 = 1$ and $x_2 = 2$. We also know that $z_3 = 1$, but the value of the slack variable does not affect the objective function. In addition, the values of all non-basic variables is 0, hence $x_3 = 0$ and the value of the objective function is 15 (as represented in the bottom right entry in the tableau).