MTAT.03.286: Advanced Methods in Algorithms **University of Tartu**

Homework 6

Due date: December 29th, 2014

It is possible to collect up to 110 points in this homework.

1. Set multicover problem is a generalization of the set cover problem. As in the set cover problem, we are given a universe set $U = \{a_1, a_2, \cdots, a_n\}$, a colection of subsets $\mathbb{S} = \{S_1, S_2, \cdots, S_k\}$, where $S_i \subseteq U$ for $i = 1, 2, \cdots, k$, and a cost function $c : \mathbb{S} \to \mathbb{Q}^+$. Additionally, for each $a \in U$, there is an integer $\tau_a > 0$, which specifies how many times a should be covered by the selected subsets. The goal is to cover all elements up to their coverage requirements at minimum total cost. It is allowed to pick the set $S_i \in \mathbb{S}$ any integer number $k \geq 0$ of times, and the cost of picking that set then becomes $k \cdot c(S_i)$.

   Propose a polynomial-time approximation algorithm for the set multicover problem, which achieves approximation factor H_n, where

   $$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} ,$$

   and prove its correctness and approximation factor.

   Hint: for example, you may use dual fitting technique.

2. Let $G(V, E)$ be a (finite) undirected graph, and let $c : V \to \mathbb{Q}^+$ be a cost function defined on the vertices in V. Consider the following LP program for solving the minimum weight vertex cover problem:

   $$\begin{aligned} \textsf{minimize} \quad & \sum_{v \in V} c(v) \cdot x_v \\ \textsf{subject to} \quad & \forall e = \{u, v\} \in E \ : \ x_u + x_v \geq 1 \\ & \forall v \in V \ : \ x_v \geq 0 \end{aligned}$$

   Here, for all $v \in V$, we defined indicator variables x_v, which are 1 if v is in the cover, and 0 otherwise.

   Show that if G is a *bipartite* graph, then all extreme point solutions of this LP problem are integral. Conclude, that LP programming can be used for solving efficiently the minimum weight vertex cover problem in a bipartite graph.

3. It is known that any (finite, undirected) planar graph is 4-colorable in polynomial time. In other words, the vertices of any planar graph can be efficiently partitioned into 4 subsets, such that there is no edge between any two vertices in the same subset.

   Show how you can use an algorithm for 4-coloring of a planar graph to find a 3/2-approximation to vertex cover in the given (finite, undirected) planar graph.

   Hint: Use the half-integrality of the vertex cover.

4. Hitting set problem is defined as follows. Given a universe set $U = \{a_1, a_2, \cdots, a_n\}$, and a collection of subsets $T_1, T_2, \cdots, T_k$, where all $T_i \subseteq U$, find a subset $M \subseteq U$ of minimum size, such that M hits every T_i, namely, $M \cap T_i \neq \varnothing$ for $i = 1, 2, \cdots, k$. Devise a primal-dual approximation algorithm for this problem. What is the approximation factor of your algorithm?

   Hint: take $\alpha = 1$. What is the value of β?