Homework 5

Due date: December 12th, 2014

It is possible to collect up to 110 points in this homework.

1. Let $G(V, E)$ be a finite strongly-connected directed graph. Let $w : E \to \mathbb{R}^+$ be a positive weight function defined on the edges of E. A vertex-disjoint cycle cover is a collection of simple circuits of length ≥ 2, such that every vertex in V participates in exactly one such circuit. Prove that the minimum weight vertex-disjoint cycle cover can be found in a polynomial time.

   Hint: it was briefly mentioned in the lecture that finding a maximum (weighted) matching in the bipartite graph requires polynomial time. You can use this fact without proving it.

   Show that finding a perfect matching of a minimum weight in a weighted bipartite graph requires polynomial time as well. Think how a vertex-disjoint cycle cover in a general directed graph can be obtained from a perfect matching in a suitable bipartite undirected graph.

2. Consider a finite complete directed graph $G(V, E)$. Let $w : E \to \{1, 2\}$ be a weight function defined on the edges in E. In other words, for every pair of vertices $u, v \in V$, the weight of the edge (u, v) is either 1 or 2.

   (a) Do the weights of the edges of G satisfy the triangle inequality? Prove or disprove.

   (b) Use the algorithm from Question 1 to give a factor $\frac{3}{2}$ approximation algorithm for the Traveling Salesman Problem for the graph G.

3. The captain of a sport team wants to place n objects $a_1, a_2, \cdots, a_n$ into $k > 1$ bags. For each $i = 1, 2, \cdots, n$, the weight of a_i is w_i kilograms. The capacity of each bag is not limited. The goal is to minimize the weight of the most heavy bag.

   Consider the following greedy algorithm. The objects are ordered in an arbitrary order. The captain always places the object under consideration into the least heavy bag. Show that at the end of the algorithm run, the weight of the most heavy bag is at most 2 times the optimum.

4. In this problem, the captain wants to place the n objects $a_1, a_2, \cdots, a_n$ into **two** bags. As in the previous question, the weight of a_i is w_i kilograms, and the capacity of each bag is not limited. The goal is to minimize the weight of the most heavy bag.

   Consider the same greedy algorithm. The objects are ordered in an arbitrary order. The captain always places the object under consideration into the least heavy bag.

   (a) Show that at the end of the algorithm run, the weight of the most heavy bag is at most $\frac{3}{2}$ times the optimum.

   (b) Show an example of the weights w_i, $i = 1, 2, \cdots, n$, such that the output of the algorithm is exactly a factor $\frac{3}{2}$ of the optimum.