MTAT.03.286: Advanced Methods in Algorithms **University of Tartu**

Homework 4

Due date: November 26, 2014

It is possible to collect up to 110 points in this homework.

1. Let $S = \{x_1, x_2, \cdots, x_n\}$ be a set of Boolean variables of size $n \geq 1$.

   Definitions (reminder):

   - An *atom* is a Boolean variable from S. For example, x_3 is an atom (when $n \geq 3$).
   - A *literal* is an atom or its negation. For example, x_5 and $\neg x_2$ are two literals.
   - A *clause* is a disjunction of literals. For example, $(\neg x_1 \vee x_3 \vee \neg x_4)$ is a clause.
   - A *conjunctive normal form (CNF) formula* is a conjunction of some clauses. For example, $(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee x_{17})$ is a CNF formula.

   Let $\phi(x_1, x_2, \cdots, x_n)$ be a CNF formula with m clauses, where each clause contains exactly k literals, and m and k are positive integers.

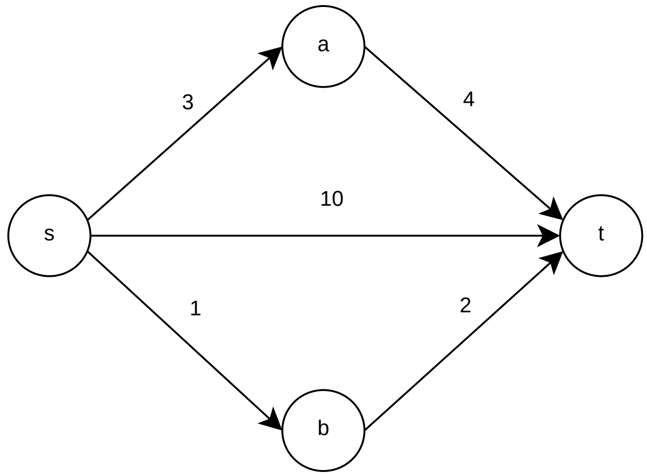
   (a) Propose a simple probabilistic algorithm that finds an assignment of True/False to variables in S, which satisfies the formula ϕ. The failure probability of the proposed algorithm should be at most $\frac{m}{2^k}$.
   Hint: if $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_m$ are random events, then (by the union bound):

   $$\mathsf{Prob}\left(\mathcal{E}_1 \cup \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_m\right) \leq \sum_{i=1}^{m} \mathsf{Prob}(\mathcal{E}_i) \ .$$

   (b) How many parallel runs of the algorithm are needed in order to make the probability of a failure at most $\frac{1}{2^t}$, where $t \geq 1$ is a parameter?

   (c) Show an example of ϕ, which has no satisfying assignment for $m = 2^k$.

2. Consider the following flow network $\mathcal{N}$.



(a) Write the problem of finding maximum flow from s to t in $\mathcal{N}$ as a linear program.

(b) Write down the dual of this linear program. There should be a dual variable for each edge of the network and for each vertex other than s and t.

Now, consider a general flow network. Recall the linear program formulation for a general maximum flow problem, which was shown in the class.

(c) Write down the dual of this general flow linear-programming problem, using a variable y_e for each edge and x_u for each vertex $u \neq s, t$.

(d) Show that any solution to the general dual problem must satisfy the following property: for any directed path from s to t in the network, the sum of y_e values along the path must be at least 1.

3. **Definition.** An *independent set* is a set of vertices in a graph, no two of which are adjacent. The *maximum independent set problem* is the problem of finding an independent set of maximum size.

   Definition. An *edge cover* of a graph is a set of edges such that every vertex of the graph is incident with at least one edge of the set. The *minimum edge cover problem* is the problem of finding an edge cover of minimum size.

   (a) Formulate maximum independent set problem as **integer** linear-programming problem.

   (b) Formulate minimum edge cover problem as **integer** linear-programming problem.

   (c) In the two formulations, relax constraints such that the resulting problems become linear-programming problems, and the resulting two problems are dual.

4. Solve the following linear-programming problem using simplex algorithm:

$$
\begin{array}{rl}
\textbf{max} & x_1 + x_2 + x_3 \\
\textbf{s.t.} & 2x_1 + x_2 \leq 5 \\
 & 2x_2 + x_3 \leq 4 \\
 & x_3 \leq 2 \\
 & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0
\end{array}
$$