MTAT.03.286: Advanced Methods in Algorithms **University of Tartu**

Final exam

Instructor: Dr. Vitaly Skachek January 13th, 2015

Student name: ________________________________

Student ID: ________________________________

1. This exam contains 10 pages. Check that no pages are missing.
2. It is possible to collect up to 110 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 3 hours.
7. Good luck!

Question 1	
Question 2	
Question 3	
Question 4	
Total	

Question 1 (20 points).

By using Fast Fourier Transform (FFT) algorithm, evaluate the polynomial $P(x) = x^3 - 2x^2 + 2x - 1$ at the complex 4th roots of unity. Show at least one level of recursion.

Question 2 (30 points). You are given the following linear-programming (LP) problem:

$$\begin{array}{rcccccl} \textbf{max} & x_1 & - & 2x_2 & + & x_3 & \\ \textbf{s.t.} & x_1 & + & x_2 & & & \leq 3 \\ & x_1 & + & & & 2x_3 & \leq 5 \\ & & & x_2 & - & x_3 & \leq 2 \\ & \multicolumn{6}{c}{x_1 \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0} \end{array}$$

(a) Solve this LP problem by using the simplex method.

(b) Formulate the dual LP problem.

(c) Verify that your solution of (a) is optimal by the substitution of the point $(\frac{1}{2}, \frac{1}{2}, 0)$ into the dual problem. Explain your answer.

Question 3 (30 points). Let $\boldsymbol{A}$ be an $n \times n$ matrix, whose entries are '0' and '1'. A *generalized diagonal* in $\boldsymbol{A}$ is a set of exactly n entries in $\boldsymbol{A}$, such that:

- Exactly one entry is selected from each row and from each column;
- Every selected entry is '1'.

Propose an algorithm that finds a generalized diagonal in a given matrix $\boldsymbol{A}$ as above. If no generalized diagonal exists, the algorithm prints a corresponding message. The required time complexity is $O(n^3)$.

Proof correctness of your algorithm and analyze its time complexity.

Question 4 (30 points).

A tourist wants to place n objects $a_1, a_2, \cdots, a_n$ into k suitcases. For each $i = 1, 2, \cdots, n$, the weight of a_i is w_i kilograms. The capacity of each suitcase is not limited. The goal is to minimize the weight of the most heavy suitcase.

The following greedy algorithm is used (it is the same algorithm as in Questions 3 and 4 of Homework 5). The objects are ordered in an arbitrary order. The tourist always places the object under consideration into the least heavy suitcase.

(a) Show that, for $k = 3$, at the end of the algorithm run, the weight of the most heavy suitcase is at most $\frac{5}{3}$ times the optimum.

(b) Generalize the result from part (a) and conclude that for general k, the weight of the most heavy suitcase is at most $2 - \frac{1}{k}$ times the optimum. (Note that it is a better approximation factor than in Question 3 of Homework 5).