MTAT.03.286: Advanced Methods in Algorithms **University of Tartu**

Homework 4

Due date: November 14, 2013

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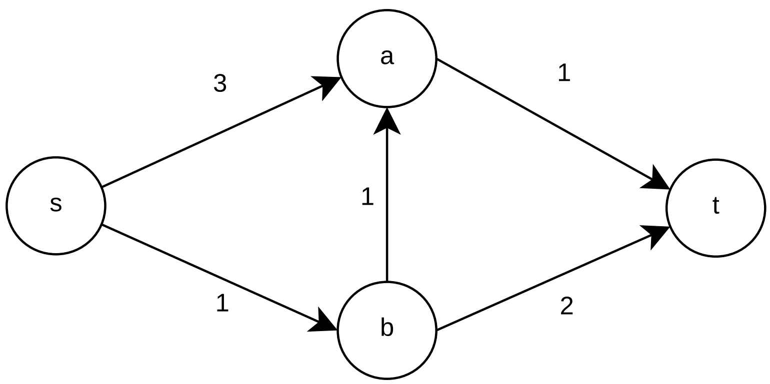
1. Consider a (bounded) linear-programming problem, which has an optimal feasible solution. Show that there exists a vertex, such that the optimum of the objective function is obtained at that vertex.

   Hint: use convexity of the feasible region. More specifically, it was shown in the class that for any two feasible points $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{y}}$, for any $\alpha \in (0,1)$, the point $\alpha \cdot \overline{\boldsymbol{x}} + (1-\alpha) \cdot \overline{\boldsymbol{y}}$ is also a feasible point. Recall that the vertices of the feasible region are obtained by setting $\geq n-1$ constraints to equalities, where n is the dimension of the solution space (number of variables).

2. Consider the following flow network $\mathcal{N}$.



   (a) Write the problem of finding maximum flow from s to t in $\mathcal{N}$ as a linear program.

   (b) Write down the dual of this linear program. There should be a dual variable for each edge of the network and for each vertex other than s and t.

   Now, consider a general flow network. Recall the linear program formulation for a general maximum flow problem, which was shown in the class.

   (c) Write down the dual of this general flow linear-programming problem, using a variable y_e for each edge and x_u for each vertex $u \neq s, t$.

   (d) Show that any solution to the general dual problem must satisfy the following property: for any directed path from s to t in the network, the sum of y_e values along the path must be at least 1.

3. **Definition.** An *independent set* is a set of vertices in a graph, no two of which are adjacent. The *maximum independent set problem* is the problem of finding an independent set of maximum size.

Definition. An *edge cover* of a graph is a set of edges such that every vertex of the graph is incident with at least one edge of the set. The *minimum edge cover problem* is the problem of finding an edge cover of minimum size.

(a) Formulate maximum independent set problem as **integer** linear-programming problem.

(b) Formulate minimum edge cover problem as **integer** linear-programming problem.

(c) In the two formulations, relax constraints such that the resulting problems become linear-programming problems, and the resulting two problems are dual.

4. Solve the following linear-programming problem using simplex algorithm:

$$
\begin{array}{rl}
\textbf{max} & 4x_1 + 5x_2 \\
\textbf{s.t.} & x_1 + x_2 + x_3 \leq 3 \\
 & 2x_1 - x_2 \leq 3 \\
 & 2x_2 - 3x_3 \leq 2 \\
 & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0
\end{array}
$$