MTAT.03.286: Advanced Methods in Algorithms **University of Tartu**

Homework 3

Due date: October 31, 2013

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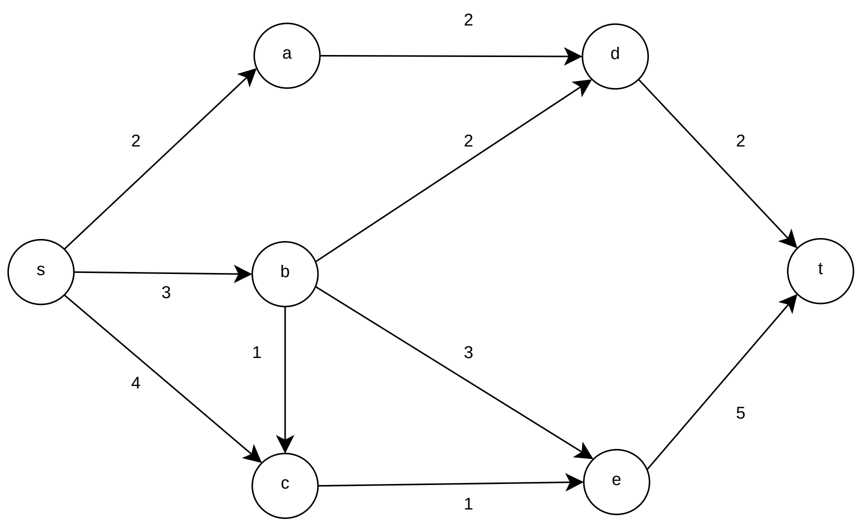
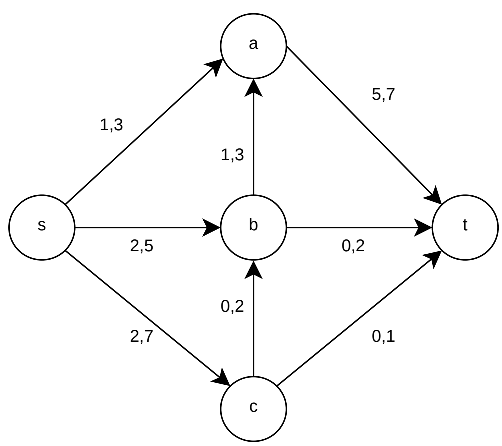
1. Find a maximum flow between s and t in the following network using Dinitz algorithm:



Show the minimum cut. How many different minimum cuts can you find?

2. (a) Find a legal flow between s and t in the following network with upper and lower bounds. (You don't have to specificy all the steps in Ford-Flukerson or Dinitz algorithm that you are using, but you have to show the reduction and the resulting flow.)



(b) Find a maximum flow in that network.

3. Let $\mathcal{H}(G(V,E),s,t,c)$ be a network, where G is a finite directed graph, and $c: E \to \mathbb{R}^+$ is a capacity function. For any two vertices $x, y \in E$, denote by $F_{x,y}$ the maximum flow in $\mathcal{N}$ when x is the source and y is the sink. Prove that for all vertices $u, v, w \in E$,

$$F_{u,w} \geq \min\{F_{u,v}, F_{v,w}\} \ .$$

4. In the Institute of Computer Science, n students work as supervisors at a computer lab. During the week, there are m different supervision time slots. All the students have submitted the lists of the slots they can work at.

   Propose an algortihm that assigns the time slots to the students, under the following conditions:

   - Each student will obtain only slots, which he included in his list.
   - At each time slot, there will be exactly three supervisors in the lab.
   - No student will be scheduled for more than seven slots on the same week.

   If there is no legal arrangement of the slots to the students, the algorithm will output an appropriate message. The required time complexity is $O(nm^2)$.

5. (Bonus question)

   In this question we will prove that if ℓ_k and ℓ_{k+1} are the number of layers in the layered network in phases k and $k+1$, respectively, of Dinitz algorithm, then $\ell_{k+1} > \ell_k$.

   We know that in phase $k+1$ there is a path P in the layered network of length ℓ_{k+1} from $s = v_0$ to $t = v_{\ell_{k+1}}$ as follows:

$$P \ : \ v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{\ell_{k+1}-1} \longrightarrow v_{\ell_{k+1}}$$

   where v_i belongs to layer i for $i = 0, 1, 2, \cdots, \ell_{k+1}$.

   (a) Assume that all vertices of P appeared also in the layered network in phase k. Show that if v_i appeared in layer j in that network, then $i \geq j$. (Hint: induction on i.)

   (b) Conclude from (a) that t appeared (in the k-th layered network) in layer j, such that $j < \ell_{k+1}$.

   (c) Explain, why it follows from (b) that $\ell_{k+1} > \ell_k$.

   (d) Now, instead of (a), assume that not all vertices of P appeared in the k-th layered network. Take the first edge of P, $v_i \to v_{i+1}$, such that v_{i+1} is not in that network. Show that this is only possible if v_i belongs to layer $\ell_k - 1$ in the k-th layered network, and $v_{i+1} \neq t$.

   (e) Conclude from (d) that $\ell_{k+1} > \ell_k$.