Simple types

Types and typing rules

Typed terms

- Types (blackboard).
- Variables and terms (blackboard).
- Type environments (blackboard).

Typing

- Typings and typing assertions (blackboard).
- Typing rules (blackboard).
- Typing derivations (blackboard).

Evaluation

- Values (blackboard).
- Reduction rules (blackboard).
- Reduction sequences (blackboard).

Examples

- Find a derivation of the type assertion

$$f : X \to X \vdash (\lambda x : X \,.\, f\, x) : X \to X$$

(blackboard).

- Let

$$\begin{aligned}
\mathsf{W} &= \lambda h : (A \to A) \to (A \to A) \to A \to A \,.\, \lambda x : A \to A \,.\, h\, x\, x,\\
\mathsf{K} &= \lambda x : A \to A \,.\, \lambda y : A \to A \,.\, x,\\
\mathsf{I} &= \lambda z : A \to A \,.\, z,\\
\mathsf{I}' &= \lambda z : A \,.\, z
\end{aligned}$$

and let $t = \mathsf{W}\,\mathsf{K}\,(\mathsf{I}\,\mathsf{I}')$. Find a type T and a derivation of the type assertion $\vdash t : T$, and a reduction sequence of t that ends with a value (blackboard).

Exercises

- Find a derivation of the type assertion $x : X \vdash (\lambda z : X \,.\, z)\, x : X$ (oneself).
- Find a type environment Γ such that $\Gamma \vdash f\, x\, y : X$ is derivable (oneself).
- Let

$$
\begin{aligned}
\mathsf{W} &= \lambda h : (A \to A) \to (A \to A) \to A \to A \,.\, \lambda x : A \to A \,.\, h\, x\, x, \\
\mathsf{K} &= \lambda x : (A \to A) \to A \to A \,.\, \lambda y : A \to A \,.\, x, \\
\mathsf{I} &= \lambda z : A \to A \,.\, z, \\
\mathsf{I}' &= \lambda z : A \,.\, z
\end{aligned}
$$

  and let $t = \mathsf{W}\,(\mathsf{K}\,\mathsf{I})\,\mathsf{I}'$. Find a type T and a derivation of the type assertion $\vdash t : T$, and a reduction sequence of t that ends with a value (blackboard).

Type erasure

Definitions

- Definition 9.5.1: erasure (blackboard).
- Definition 9.5.3: typability (blackboard).

Theorems

- Theorem 9.5.2: Erasure preserves reduction (blackboard).
- Exercise 8.2.3: Subterms of typable terms are typable (oneself).
- Exercise 9.3.2: The term $x\,x$ is not typable (oneself).

Properties of typing

Basic lemmas

- **Weakening**: If all typings of Γ occur in Δ then derivability of $\Gamma \vdash t : T$ implies derivability of $\Delta \vdash t : T$ (blackboard).
- **Permutation**: If Γ is a type environment and Δ is its permutation then derivability of $\Gamma \vdash t : T$ is equivalent to derivability of $\Delta \vdash t : T$ for every term t and type T (blackboard).
- **Substitution**: If both $\Gamma, x : S \vdash t : T$ and $\Gamma \vdash s : S$ are derivable then $\Gamma \vdash [x \mapsto s]\ t : T$ is derivable (blackboard).

Uniqueness

- Let Γ be a type environment and t a term. Then there exists at most one type T such that $\Gamma \vdash t : T$ is derivable (blackboard).
- Let Γ be a type environment, t a term and T a type. Then there exists at most one derivation of $\Gamma \vdash t : T$ (blackboard).

Safety

- **Progress**: For any term t and type T, if $\vdash t : T$ then either t is a value or $t \Rightarrow_\beta u$ for some term u (blackboard).
- **Preservation**: For arbitrary type environment Γ, terms t, u and type T, if $t \Rightarrow_\beta u$ and $\Gamma \vdash t : T$ is derivable then $\Gamma \vdash u : T$ is derivable (blackboard).

Subject reduction and expansion

- Preservation is often called **subject reduction** property because t and T in the typing $t : T$ are linguistically a subject and a predicate whence the property states preservation of typing under reduction of the subject (blackboard).
- Does the **subject expansion** property also hold: For arbitrary type environment Γ, terms t, u and type T, if $t \Rightarrow_{\beta} u$ and $\Gamma \vdash u : T$ is derivable then $\Gamma \vdash t : T$ is derivable (oneself)?

Normalization

- Exercise 12.1.1 (oneself).
- Definition 12.1.2 detailed (blackboard).
- Lemma 12.1.4 (blackboard).
- Lemma 12.1.5 (blackboard).
- Theorem 12.1.6: If $\vdash t : T$ is derivable then t is normalizable (blackboard).

Curry-Howard Isomorphism

Propositional logic with implication

- Propositions (blackboard).
- Sequents (blackboard).
- Derivations (blackboard).

Correspondence between type system and logic

- Types vs propositions, terms vs proofs (blackboard).
- Cut elimination (blackboard).