Algorithmics (6EAP)

Full text indexing
Suffix trees

Jaak Vilo
2020 fall
Content: Full-text indexing

• Suffix trie, **Suffix tree**
  – Application examples

• **Suffix array**

• Burrows Wheeler Transform (BWT)
  – Compact Suffix tree representation: BWT + succinct
Problem

• Given P and S – find all exact (or approximate) occurrences of P in S

• You are allowed to preprocess S (and P, of course)

• Goal: to speed up the searches
E.g. Dictionary problem

• Does $P$ belong to a dictionary $D=\{d_1,\ldots,d_n\}$
  – Build a binary search tree of $D$
  – B-Tree of $D$
  – Hashing
  – Sorting + Binary search

• Build a keyword trie: search in $O(|P|)$
  – Assuming alphabet has up to a constant size $c$
  – See Aho-Corasick algorithm, Trie construction
Sorted array and binary search
Sorted array and binary search

\[ O( |P| \log n ) \]
Trie for $D=\{\text{he, hers, his, she}\}$

$$O( |P| )$$
S is not a set of words, but a single long string

- S of length n
- How to index?

- Index **from every position** of a text
- **Prefix of every possible suffix** is important
Trie(babaab)

babaab
abaab
baab
aab
ab
b
**Suffix tree**

- **Definition:** A compact representation of a trie corresponding to the suffixes of a given string where all nodes with one child are merged with their parents.
- **Definition (suffix tree).** A suffix tree $T$ for a string $S$ (with $n = |S|$) is a rooted, labeled tree with a leaf for each non-empty suffix of $S$. Furthermore, a suffix tree satisfies the following properties:
  - Each internal node, other than the root, has at least two children;
  - Each edge leaving a particular node is labeled with a non-empty substring of $S$ of which the first symbol is unique among all first symbols of the edge labels of the edges leaving this particular node;
  - For any leaf in the tree, the concatenation of the edge labels on the path from the root to this leaf exactly spells out a non-empty suffix of $S$.
Literature on suffix trees

Partly based on:

**Suffix tree and suffix array techniques for pattern analysis in strings**  
*Esko Ukkonen*  
*Univ Helsinki*  
**Erice School 30 Oct 2005**  

**High-throughput genome-scale sequence analysis and mapping using compressed data structures**  
*Veli Mäkinen*  
Department of Computer Science  
University of Helsinki
ttttttttttttttttagacggaga\text{gtctcgctctgctgctccagggcttcgctctcctgctgcagttgccgcagcttcaagcttccgccttcgcacgccatctgcccttcctgcactcgtgtcggccttcctgcactcagccctacgcgtccctgcttc\text{cagctgcagggggacggggtttcaccggttttagcctgggattacaggcgt}}

1. **Suffix tree**

2. **Suffix array**

3. **Some applications**

4. **Finding motifs**

The suffix tree $\text{Tree}(T)$ of $T$

- data structure **suffix tree**, $\text{Tree}(T)$, is a **compacted trie** that represents all the suffixes of string $T$
- linear size: $|\text{Tree}(T)| = O(|T|)$
- can be constructed in linear time $O(|T|)$
- has **myriad virtues** (A. Apostolico)
- is well-known: 578 000 Google hits

Suffix trie and suffix tree

abaab
baab
aab
ab
b

Trie(abaab)

Suffix trie and suffix tree

abaab
baab
aab
ab
b

Trie(abaab)

Tree(abaab)
Trie(T) can be large

- $|\text{Trie}(T)| = O(|T|^2)$
- bad example: $T = a^n b^n$
- Trie(T) can be seen as a DFA: language accepted = the suffixes of T
- minimize the DFA => directed cyclic word graph (’DAWG’)


Tree(T) is of linear size

- only the internal branching nodes and the leaves represented explicitly
- edges labeled by substrings of T
- $v = \text{node}(\alpha)$ if the path from root to $v$ spells $\alpha$
- one-to-one correspondence of leaves and suffixes
- $|T|$ leaves, hence $< |T|$ internal nodes
- $|\text{Tree}(T)| = O(|T| + \text{size(edge labels)})$
Tree(hattivatti)
Tree(hattivatti)

substring labels of edges represented as pairs of pointers
Tree(hattivatti)
Tree(T) is full text index

P occurs in T at locations 8, 31, ...

P occurs in T ⇔ P is a prefix of some suffix of T
⇔ Path for P exists in Tree(T)

All occurrences of P in time $O(|P| + \#occ)$
Find att from Tree(hattivatti)
Linear time construction of Tree(T)

Weiner (1973), 'algorithm of the year'
McCreight (1976)

'on-line' algorithm (Ukkonen 1992)
On-line construction of Trie(T)

• \( T = t_1 t_2 \ldots t_n \$
• \( P_i = t_1 t_2 \ldots t_i \) \( i:th \) prefix of \( T \)
• **on-line idea**: update \( Trie(P_i) \) to \( Trie(P_{i+1}) \)
• => very simple construction
Trie(abaab)

Trie(a) Trie(ab) Trie(aba)

aba a  
  baa a

aa  
 εa  
 ε

chain of links connects the end points of current suffixes
Trie(abaab)
Trie(abaab)

Add next symbol = b

Trie(abaa)

Add next symbol = b
Trie(abaab)

Add next symbol = b

From here on b-arc already exists
Trie(abaab)
What happens in \( \text{Trie}(P_i) \Rightarrow \text{Trie}(P_{i+1}) \) ?

Before

From here on the \( a_i \)-arc exists already => stop updating here

After

New suffix links

New nodes
What happens in $\text{Trie}(P_i) \Rightarrow \text{Trie}(P_{i+1})$?

- time: $O(\text{size of Trie}(T))$
- suffix links:
  \[
  \text{slink}(\text{node}(a\alpha)) = \text{node}(\alpha)
  \]
On-line procedure for suffix trie

1. Create $\text{Trie}(t_1)$: nodes $\text{root}$ and $v$, an arc $\text{son}(\text{root}, t_1) = v$, and suffix links $\text{slink}(v) := \text{root}$ and $\text{slink}(\text{root}) := \text{root}$

2. for $i := 2$ to $n$ do begin

3. $v_{i-1} :=$ leaf of $\text{Trie}(t_1...t_{i-1})$ for string $t_1...t_{i-1}$ (i.e., the deepest leaf)

4. $v := v_{i-1}$; $v' := 0$

5. while node $v$ has no outgoing arc for $t_i$ do begin

6. Create a new node $v''$ and an arc $\text{son}(v, t_i) = v''$

7. if $v' \neq 0$ then $\text{slink}(v) := v''$

8. $v := \text{slink}(v)$; $v' := v''$ end

9. for the node $v''$ such that $v'' = \text{son}(v, t_i)$ do

   if $v'' = v'$ then $\text{slink}(v') := \text{root}$ else $\text{slink}(v') := v''$
Suffix trees on-line

• ‘compacted version’ of the on-line trie construction: simulate the construction on the linear size tree instead of the trie => time $O(|T|)$

• all trie nodes are conceptually still needed => *implicit* and *real* nodes
Implicit and real nodes

• Pair \((v, \alpha)\) is an *implicit node* in Tree(T) if \(v\) is a node of Tree and \(\alpha\) is a (proper) prefix of the label of some arc from \(v\). If \(\alpha\) is the empty string then \((v, \alpha)\) is a *real* node (= \(v\)).

• Let \(v = node(\alpha \hat{\cdot})\) in Tree(T). Then implicit node \((v, \alpha)\) represents \(node(\alpha \hat{\cdot} \alpha)\) of Trie(T)
Implicit node

\[ (v, \alpha) \]

\[ \alpha \]

\[ \alpha^\prime \]
Suffix links and open arcs

α

... root

slink(v)

... 

label [i,*] instead of [i,j] if w is a leaf and j is the scanned position of T
suffix link path traversed: total work $O(n)$

new arcs and nodes created: total work $O(\text{size(Tree(T))})$
On-line procedure for suffix tree

Input: string $T = t_1 t_2 \ldots t_n$

Output: $Tree(T)$

Notation: $son(v, \alpha) = w$ iff there is an arc from $v$ to $w$ with label $\alpha$

$son(v, \varepsilon) = v$

Function $Canonize(v, \alpha)$:

\[
\textbf{while } son(v, \alpha{'}) \neq 0 \text{ where } \alpha = \alpha{'}, |\alpha| > 0 \text{ \ do } \\
\quad v := son(v, \alpha{'}); \quad \alpha := \alpha{''} \\
\textbf{return } (v, \alpha)
\]
Suffix-tree on-line: main procedure

Create $Tree(t_1)$; $slink(root) := root$

$(v, \alpha) := (root, \varepsilon)$  /* $(v, \alpha)$ is the start node */

for $i := 2$ to $n+1$ do

$v^\prime := 0$

while there is no arc from $v$ with label prefix $at_i$ do

if $\alpha \neq \varepsilon$ then /* divide the arc $w = son(v, \alpha \eta)$ into two */

$son(v, \alpha) := v^\prime$; $son(v^\prime, t_i) := v^\prime^\prime$; $son(v^\prime^\prime, \eta) := w$

else

$son(v, t_i) := v^\prime^\prime$; $v^\prime^\prime := v$

if $v^\prime \neq 0$ then $slink(v^\prime) := v^\prime^\prime$

$v^\prime := v^\prime^\prime$; $v := slink(v)$; $(v, \alpha) := Canonize(v, \alpha)$

if $v^\prime \neq 0$ then $slink(v^\prime) := v$

$(v, \alpha) := Canonize(v, at_i)$  /* $(v, \alpha)$ = start node of the next round */
Example

...text indexed is here as an example. here is there is Jeremy...
Applications of Suffix Trees


- **APL1**: Exact String Matching Search for P from text S. Solution 1: build STree(S) - one achieves the same O(n+m) as Knuth-Morris-Pratt, for example!
- Search from the suffix tree is O(|P|)
- **APL2**: Exact set matching Search for a set of patterns P
Applications of SuffixTrees

- **APL3**: substring problem for a database of patterns
  Given a set of strings \(S=S_1, \ldots, S_n\) --- a database Find all \(S_i\) that have \(P\) as a substring
- Generalized suffix tree contains all suffixes of all \(S_i\)
- Query in time \(O(|P|)\), and can identify the LONGEST common prefix of \(P\) in all \(S_i\)
Applications of Suffix Trees

- **APL4**: Longest common substring of two strings
- Find the longest common substring of S and T.
- Overall there are potentially $O(n^2)$ such substrings, if $n$ is the length of a shorter of S and T
- Donald Knuth once (1970) conjectured that linear-time algorithm is impossible.
- Solution: construct the STree(S+T) and find the node deepest in the tree that has suffixes from both S and T in subtree leaves.
- Ex: S= superiorcalifornialives T= sealiver have both a substring *alive*. 
Simple analysis task: LCSS

• Let $\text{LCSSA}(A,B)$ denote the longest common substring two sequences $A$ and $B$. E.g.:
  – $\text{LCSS}(\text{AGATCTATCT}, \text{CGCCTCTATG}) = \text{TCTAT}$.

• A good solution is to build suffix tree for the shorter sequence and make a descending suffix walk with the other sequence.
Suffix link
Descending suffix walk

Read B left-to-right, always going down in the tree when possible. If the next symbol of B does not match any edge label on current position, take suffix link, and try again. (Suffix link in the root to itself emits a symbol). The node $v$ encountered with largest string depth is the solution.
Applications of Suffix Trees

- **APL5**: Recognizing DNA contamination Related to DNA sequencing, search for longest strings (longer than threshold) that are present in the DB of sequences of other genomes.
- **APL6**: Common substrings of more than two strings Generalization of APL4, can be done in linear (in total length of all strings) time
Another common tool: Generalized suffix tree

node info:
subtree size 47813871
sequence count 87

ACCTTA....ACCT#CACATT..CAT#TGTCGT...GTA#TCACCACC...C$
Generalized suffix tree application

...ACC...#...ACC...#...ACC...ACC..ACC..#..ACC..ACC...#...ACC...#...

...#....#...#...#...#...ACC...#...#...#...#...#...#..#..ACC..ACC...#......#...

node info:
subtree size 4398
blue sequences 12/15
red sequences 2/62
Applications of Suffix Trees

• **APL7**: Building a directed graph for exact matching: **Suffix graph** - directed acyclic word graph (DAWG), a **smallest finite state automaton** recognizing all suffixes of a string S. This automaton can recognize membership, but not tell which suffix was matched.

• Construction: merge isomorphic subtrees.

• Isomorphic in Suffix Tree when exists suffix link path, and subtrees have equal nr. of leaves.
Applications of Suffix Trees

• APL8: A reverse role for suffix trees, and major space reduction Index the pattern, not tree...
• Matching statistics.
• APL10: All-pairs suffix-prefix matching For all pairs $S_i, S_j$, find the longest matching suffix-prefix pair. Used in shortest common superstring generation (e.g. DNA sequence assembly), EST alignmentm etc.
Applications of Suffix Trees

- APL11: Finding all maximal repetitive structures in linear time
- APL12: Circular string linearization e.g. circular chemical molecules in the database, one wants to linearize them in a canonical way...
- APL13: Suffix arrays - more space reduction will touch that separately
Applications of Suffix Trees

• APL14: Suffix trees in genome-scale projects
• APL15: A Boyer-Moore approach to exact set matching
• APL16: Ziv-Lempel data compression
• APL17: Minimum length encoding of DNA
Applications of Suffix Trees

• Additional applications Mostly exercises...
• Extra feature: **CONSTANT time lowest common ancestor retrieval (LCA)**
  Andmestruktuur mis võimaldab leida konstantse ajaga alumist ühist vanemat (see vastab pikimale ühisele prefixile!) on võimalik koostada lineaarse ajaga.
• APL: Longest common extension: a bridge to inexact matching
• APL: **Finding all maximal palindromes in linear time**
  Palindrome reads from central position the same to left and right. E.g.: kirik, saippuakivikauppias.
• Build the suffix tree of S and inverted S (aabcbad => aabcbad#dabcbaa ) and using the LCA one can ask for any position pair (i, 2i-1), the longest common prefix in constant time.
• The whole problem can be solved in O(n).
Applications of Suffix Trees

• APL: Exact matching with wild cards
• APL: The k-mismatch problem
• Approximate palindromes and repeats
• Faster methods for tandem repeats
• A linear-time solution to the multiple common substring problem
• And many-many more ...
Properties of suffix tree

• Suffix tree has $n$ leaves and at most $n-1$ internal nodes, where $n$ is the total length of all sequences indexed.

• Each node requires constant number of integers (pointers to first child, sibling, parent, text range of incoming edge, statistics counters, etc.).

• Can be constructed in linear time.
Properties of suffix tree... in practice

• Huge overhead due to pointer structure:
  – Standard implementation of suffix tree for human genome requires over 200 GB memory!
  – A careful implementation (using $\log n$ -bit fields for each value and array layout for the tree) still requires over 40 GB.
  – Human genome itself takes less than 1 GB using 2-bits per bp.
Algorithmics (6EAP)

Full text indexing

Suffix arrays

Jaak Vilo

2020 fall
1. Suffix tree

2. **Suffix array**

3. Some applications

4. Finding motifs
Sort all suffixes. Allows to perform binary search!
# Suffix array: example

<table>
<thead>
<tr>
<th>1</th>
<th>hattivatti</th>
<th>ε</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>attivatti</td>
<td>atti</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>ttivatti</td>
<td>attivatti</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>tivatti</td>
<td>hattivatti</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>ivatti</td>
<td>i</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>vatti</td>
<td>ivatti</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>atti</td>
<td>ti</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>tti</td>
<td>tivatti</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>ti</td>
<td>tti</td>
<td>8</td>
</tr>
<tr>
<td>10</td>
<td>i</td>
<td>ttivatti</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>ε</td>
<td>vatti</td>
<td>6</td>
</tr>
</tbody>
</table>

- suffix array = lexicographic order of the suffixes
Suffix array construction: sort!

<table>
<thead>
<tr>
<th>Suffix</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>hattivatti</td>
<td>1</td>
</tr>
<tr>
<td>attivatti</td>
<td>2</td>
</tr>
<tr>
<td>ttivatti</td>
<td>3</td>
</tr>
<tr>
<td>tivatti</td>
<td>4</td>
</tr>
<tr>
<td>ivatti</td>
<td>5</td>
</tr>
<tr>
<td>vatti</td>
<td>6</td>
</tr>
<tr>
<td>atti</td>
<td>7</td>
</tr>
<tr>
<td>tti</td>
<td>8</td>
</tr>
<tr>
<td>ti</td>
<td>9</td>
</tr>
<tr>
<td>i</td>
<td>10</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>11</td>
</tr>
</tbody>
</table>

• suffix array = lexicographic order of the suffixes
Suffix array

• **suffix array** $SA(T) = \text{an array giving the lexicographic order of the suffixes of } T$

• space requirement: $5 |T|$

• practitioners like suffix arrays (simplicity, space efficiency)

• theoreticians like suffix trees (explicit structure)
Reducing space: suffix array

C A T A C T
1 2 3 4 5 6
Suffix array

• Many algorithms on suffix tree can be simulated using *suffix array*...
  – ... and couple of additional arrays...
  – ... forming so-called *enhanced suffix array*...
  – ... leading to the similar space requirement as careful implementation of suffix tree
• Not a satisfactory solution to the space issue.
Pattern search from suffix array

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>hattivatti</td>
<td>11</td>
</tr>
<tr>
<td>attivatti</td>
<td>7</td>
</tr>
<tr>
<td>ttivatti</td>
<td>2</td>
</tr>
<tr>
<td>tivatti</td>
<td>1</td>
</tr>
<tr>
<td>ivatti</td>
<td>10</td>
</tr>
<tr>
<td>vatti</td>
<td>5</td>
</tr>
<tr>
<td>atti</td>
<td>9</td>
</tr>
<tr>
<td>tti</td>
<td>4</td>
</tr>
<tr>
<td>ti</td>
<td>8</td>
</tr>
<tr>
<td>i</td>
<td>3</td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td>6</td>
</tr>
</tbody>
</table>
Historically first?


• PAT index for Oxford English Dictionary project on CD-s
What we learn today?

• We learn that it is possible to replace suffix trees with \textit{compressed suffix trees} that take \textbf{8.8 GB} for the human genome.

• We learn that \textit{backtracking} can be done using \textit{compressed suffix arrays} requiring only \textbf{2.1 GB} for the human genome.

• We learn that \textit{discovering} interesting motif seeds from the human genome takes \textbf{40 hours} and requires \textbf{9.3 GB} space.
Recent suffix array constructions

• **Manber&Myers** (1990): $O(|T| \log |T|)$

• linear time *via suffix tree*

• January / June 2003: **direct linear time construction of suffix array**
  - Kim, Sim, Park, Park (CPM03)
  - Kärkkäinen & Sanders (ICALP03)
  - Ko & Aluru (CPM03)
A Source Code

The following C++ file contains a complete linear time implementation of suffix array construction. This code strives for conciseness rather than speed — it has only 50 lines not counting comments, empty lines, and lines with a bracket only. A driver program can be found at http://wwwmpi-sb-mpg.de/sanders/programs/suffix/.

```cpp
inline bool leq(int a1, int a2, int b1, int b2) // lexicographic order
{ return(a1 < b1 || a1 == b1 && a2 <= b2); } // for pairs
inline bool leq(int a1, int a2, int a3, int b1, int b2, int b3) // and triples
{ return(a1 < b1 || a1 == b1 && leq(a2,a3,b2,b3)); } // and triples

// stably sort a[0..n-1] to b[0..n-1] with keys in 0..K from r
static void radixPass(int* a, int* b, int* r, int n, int K)
{ // count occurrences
  int* c = new int[K+1]; // counter array
  for (int i = 0; i < K; i++) // reset counters
    c[i] = 0;
  for (int i = 0; i < n; i++) // exclusive prefix sums
    c[r[a[i]]] = c[r[a[i]]] + 1;
  for (int i = 0; i < n; i++) // reverse count
    c[r[a[i]]] = c[r[a[i]]] - 1;
  for (int i = 0; i < n; i++) b[c[r[a[i]]]] = a[i]; // sort
}
```
Kärkkäinen-Sanders algorithm

1. Construct the suffix array of the suffixes starting at positions $i \mod 3 \neq 0$. This is done by reduction to the suffix array construction of a string of two thirds the length, which is solved recursively.

2. Construct the suffix array of the remaining suffixes using the result of the first step.

3. Merge the two suffix arrays into one.
Notation

- string $T = T[0,n) = t_0 t_1 \ldots t_{n-1}$
- suffix $S_i = T[i,0) = t_i t_{i+1} \ldots t_{n-1}$
- for $C \subset [0,n)$: $S_C = \{S_i | i \text{ in } C\}$

- **suffix array** $SA[0,n]$ of $T$ is a permutation of $[0,n]$ satisfying $S_{SA[0]} < S_{SA[1]} < \ldots < S_{SA[n]}$
Running example

0  1  2  3  4  5  6  7  8  9  10  11

• $T[0,n) = y a b b a d a b b a d o 0 0 ...$

• $SA = (12,1,6,4,9,3,8,2,7,5,10,11,0)$
Step 0: Construct a sample

- for k = 0, 1, 2
  \[ B_k = \{i \in [0, n] \mid i \mod 3 = k\} \]

- \( C = B_1 \cup B_2 \)  \textit{sample positions}

- \( S_C \)  \textit{sample suffixes}

- Example: \( B_1 = \{1, 4, 7, 10\}, B_2 = \{2, 5, 8, 11\}, C = \{1, 4, 7, 10, 2, 5, 8, 11\} \)
Step 1: Sort sample suffixes

- for \( k = 1, 2 \), construct

\[
R_k = [t_{k} t_{k+1} t_{k+2}] [t_{k+3} t_{k+4} t_{k+5}] ... [t_{\text{maxB}_{k}} t_{\text{maxB}_{k+1}} t_{\text{maxB}_{k+2}}]
\]

\[
R = R_1 \, ^\wedge \, R_2 \quad \text{concatenation of } R_1 \text{ and } R_2
\]

Suffixes of \( R \) correspond to \( S_C \): suffix \([t_i t_{i+1} t_{i+2}]...\) corresponds to \( S_i \); correspondence is order preserving.

Sort the suffixes of \( R \): radix sort the characters and rename with ranks to obtain \( R' \). If all characters different, their order directly gives the order of suffixes. Otherwise, sort the suffixes of \( R' \) using Kärkkäinen-Sanders. Note: \( |R'| = 2n/3 \).
Step 1 (cont.)

- once the sample suffixes are sorted, assign a rank to each: \( \text{rank}(S_i) = \text{the rank of } S_i \text{ in } S_C; \text{rank}(S_{n+1}) = \text{rank}(S_{n+2}) = 0 \)

- Example:

  \[ R = [\text{abb}][\text{ada}][\text{bba}][\text{do0}][\text{bba}][\text{dab}][\text{bad}][\text{o00}] \]
  \[ R' = (1,2,4,6,4,5,3,7) \]
  \[ SA_{R'} = (8,0,1,6,4,2,5,3,7) \]
  \[ \text{rank}(S_i) - 1 \ 4 \ - 2 \ 6 \ - 5 \ 3 \ - 7 \ 8 \ - 0 \ 0 \]
Step 2: Sort nonsample suffixes

• for each non-sample \( S_i \in S_{B0} \) (note that \( \text{rank}(S_{i+1}) \) is always defined for \( i \in B0 \)):
  \[
  S_i \leq S_j \iff (t_i, \text{rank}(S_{i+1})) \leq (t_j, \text{rank}(S_{j+1}))
  \]
• radix sort the pairs \((t_i, \text{rank}(S_{i+1}))\).

• Example: \( S_{12} < S_6 < S_9 < S_3 < S_0 \) because \((0,0) < (a,5) < (a,7) < (b,2) < (y,1)\)
Step 3: Merge

• merge the two sorted sets of suffixes using a standard comparison-based merging:

• to compare $S_i \in S_C$ with $S_j \in S_{B0}$, distinguish two cases:

  • $i \in B1$: $S_i \leq S_j \iff (t_i, \text{rank}(S_{i+1})) \leq (t_j, \text{rank}(S_{j+1}))$
  • $i \in B2$: $S_i \leq S_j \iff (t_i, t_{i+1}, \text{rank}(S_{i+2})) \leq (t_j, t_{j+1}, \text{rank}(S_{j+2}))$

• note that the ranks are defined in all cases!

• $S_1 < S_6$ as $(a,4) < (a,5)$ and $S_3 < S_8$ as $(b,a,6) < (b,a,7)$
Running time $O(n)$

- excluding the recursive call, everything can be done in linear time
- the recursion is on a string of length $2n/3$
- thus the time is given by recurrence
  \[ T(n) = T(2n/3) + O(n) \]
- hence $T(n) = O(n)$
Implementation

- about 50 lines of C++
- code available e.g. via Juha Kärkkäinen’s home page
LCP table

• Longest Common Prefix of successive elements of suffix array:
  • LCP[i] = length of the longest common prefix of suffixes $S_{SA[i]}$ and $S_{SA[i+1]}$
  • build inverse array $SA^{-1}$ from SA in linear time
  • then LCP table from $SA^{-1}$ in linear time (Kasai et al, CPM2001)
• **Example - Word of the Day, Fourth**  
  [http://www.oed.com/cgi/display/wotd](http://www.oed.com/cgi/display/wotd)
• **PAT index** - by **Gaston Gonnet** (ta on samuti Maple tarkvara üks loojatest ning hiljem molekulaarbioloogia tarkvarapaketi väljatöötajaid)
• PAT index is essentially a suffix array. To save space, indexed only from first character of every word
• XML-tagging (or SGML, at that time!) also indexed
• To mark certain fields of XML, the bit vectors were used.
• Main concern - improve the speed of search on the CD - minimize random accesses.
• For slow medium even 15-20 accesses is too slow...
Suffix tree vs suffix array

• suffix tree $\Leftrightarrow$ suffix array + LCP table
1. Suffix tree
2. Suffix array
3. Some applications
4. Finding motifs
Substring motifs of string T

• string $T = t_1 \ldots t_n$ in alphabet $A$.

• Problem: what are the frequently occurring (ungapped) substrings of $T$? Longest substring that occurs at least $q$ times?

• Thm: Suffix tree $Tree(T)$ gives complete occurrence counts of all substring motifs of $T$ in $O(n)$ time (although $T$ may have $O(n^2)$ substrings!)
Counting the substring motifs

- internal nodes of Tree(T) ↔ repeating substrings of T
- number of leaves of the subtree of a node for string P = number of occurrences of P in T
Substring motifs of hattivatti

Counts for the $O(n)$ maximal motifs shown
Finding repeats in DNA

- human chromosome 3
- the first 48 999 930 bases
- 31 min cpu time (8 processors, 4 GB)

- Human genome: $3 \times 10^9$ bases
- Tree(HumanGenome) feasible
Longest repeat?

**Occurrences at:** 28395980, 28401554r  
**Length:** 2559

ttaggtacatgtgcacaacagtgcaggttggctgctgcaccataactctctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaagacatgacacactcatttttattggtgctgcataagtctccctatattagcgggtttttgttcatatgtatataactcatgtatacaggtgcagggtgttgcaggttttttatagtctccgaattgctatccctcccccctcccccccacccccccacagtgcaccaggttggtgtgctgcaccctatctatccctaaaагtagttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttt
Ten occurrences?

```
tttttttttttttgagacggagttctcgctctgtgcccaggctggagtgcagttggcgggatctcggctcactgcaagctccgcctcccgggttcacgccattctcctgcctcagcctcccaagtagctgggactacaggcgcccgccactacgcccggctaattttttgtatttttagt
tagacgcgggttttcaccgttttagcgggattgtctctgatctcctagctccgtgatccgcccgcctcggcctcccaaagtgctgggattacaggcgt
```

**Length:** 277

**Occurrences** at: 10130003, 11421803, 18695837, 26652515, 42971130, 47398125

In the reversed complement at: 17858493, 41463059, 42431718, 42580925
Using suffix trees: plagiarism

• find longest common substring of strings X and Y
• build Tree(X$Y) and find the deepest node which has a leaf pointing to X and another pointing to Y
Using suffix trees: approximate matching

- edit distance: insertions, deletions, changes

- STOCKHOLM vs TUKHOLMA
String distance/similarity functions

STOCKHOLM vs TUKHOLMA

STOCKHOLM__TU__KHOLMA

=> 2 deletions, 1 insertion, 1 change
Dynamic programming

\[ d_{i,j} = \min ( \begin{cases} 
    a_i = b_j & \text{then } d_{i-1,j-1} \\
    d_{i-1,j} + 1, \\
    d_{i,j-1} + 1 
\end{cases} ) \]

= distance between i-prefix of A and j-prefix of B
(substitution excluded)
\[ d_{i,j} = \min \begin{cases} \text{if } a_i = b_j \text{ then } d_{i-1,j-1} \text{ else } \infty, \\
  d_{i-1,j} + 1, \\
  d_{i,j-1} + 1 \end{cases} \]

<table>
<thead>
<tr>
<th>A \backslash B</th>
<th>s</th>
<th>t</th>
<th>o</th>
<th>c</th>
<th>k</th>
<th>h</th>
<th>o</th>
<th>l</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>t</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>u</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>k</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>h</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>o</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>l</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>m</td>
<td>7</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>a</td>
<td>8</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
</tr>
</tbody>
</table>

optimal alignment by trace-back

\[ d_{ID}(A,B) \]
Search problem

- find approximate occurrences of pattern \( P \) in text \( T \): substrings \( P' \) of \( T \) such that \( d(P, P') \) small
- dyn progr with small modification: \( O(mn) \)
- lots of (practical) improvement tricks
Index for approximate searching?

- dynamic programming: P x Tree(T) with backtracking
Algorithmics (6EAP)

Burrows-Wheeler transformation

Jaak Vilo
2020 fall
Burrows-Wheeler Transformation

- BWT for text compression and indexing

<table>
<thead>
<tr>
<th>Input</th>
<th>SIX.MIXED.PIXIES.SIFT.SIXTY.PIXIE.DUST.BOXES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>TEXYDST.E.IXIXIXXSUMPPS.B..E.S.EUSFXDIIIOIIIT</td>
</tr>
</tbody>
</table>
Burrows-Wheeler

- The method described in the original paper is really a composite of three different algorithms:
  - the block sorting main engine (a lossless, very slightly expansive preprocessor),
  - the move-to-front coder (a byte-for-byte simple, fast, locally adaptive noncompressive coder) and
  - a simple statistical compressor (first order Huffman is mentioned as a candidate) eventually doing the compression.
- Of these three methods only the first two are discussed here as they are what constitutes the heart of the algorithm. These two algorithms combined form a completely reversible (lossless) transformation that - with typical input - skews the first order symbol distributions to make the data more compressible with simple methods. Intuitively speaking, the method transforms slack in the higher order probabilities of the input block (thus making them more even, whitening them) to slack in the lower order statistics. This effect is what is seen in the histogram of the resulting symbol data.
- Please, read the article by Mark Nelson:
<table>
<thead>
<tr>
<th>String 0</th>
<th>D</th>
<th>R</th>
<th>D</th>
<th>O</th>
<th>B</th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>R</td>
<td>D</td>
<td>O</td>
<td>B</td>
<td>B</td>
<td>S</td>
<td>D</td>
</tr>
<tr>
<td>S2</td>
<td>D</td>
<td>O</td>
<td>B</td>
<td>B</td>
<td>S</td>
<td>D</td>
<td>R</td>
</tr>
<tr>
<td>S3</td>
<td>O</td>
<td>B</td>
<td>B</td>
<td>S</td>
<td>D</td>
<td>R</td>
<td>D</td>
</tr>
<tr>
<td>S4</td>
<td>B</td>
<td>B</td>
<td>S</td>
<td>D</td>
<td>R</td>
<td>D</td>
<td>O</td>
</tr>
<tr>
<td>S5</td>
<td>B</td>
<td>S</td>
<td>D</td>
<td>R</td>
<td>D</td>
<td>O</td>
<td>B</td>
</tr>
<tr>
<td>S6</td>
<td>S</td>
<td>D</td>
<td>R</td>
<td>D</td>
<td>O</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>
BWT
Column L, start from 6

How so we decode the message? What is the next letter?
<table>
<thead>
<tr>
<th>O</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>R</td>
<td>D</td>
</tr>
<tr>
<td>S</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>O</td>
</tr>
<tr>
<td>D</td>
<td>R</td>
</tr>
<tr>
<td>B</td>
<td>S</td>
</tr>
</tbody>
</table>

Next letter is the F column, where characters are in alphabetic order
<table>
<thead>
<tr>
<th>O</th>
<th>B</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>B</td>
<td>7</td>
</tr>
<tr>
<td>R</td>
<td>D</td>
<td>5</td>
</tr>
<tr>
<td>S</td>
<td>D</td>
<td>6</td>
</tr>
<tr>
<td>D</td>
<td>O</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>R</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>S</td>
<td>4</td>
</tr>
</tbody>
</table>

First B in F is first B in L

Letters follow stable sort

6 = D
<table>
<thead>
<tr>
<th></th>
<th>O</th>
<th>B</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>B</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>D</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>D</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>O</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>R</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>S</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

$6 = D$

$6351274$

$DRDOBBS$
CODE:

t: hat acts like this:

t: hat buffer to the constructor

t: hat corrupted the heap, or wo

W: hat goes up must come down

t: hat happens, it isn't likely

w: hat if you want to dynamically

t: hat indicates an error.

t: hat it removes arguments from

t: hat looks like this:

t: hat looks something like this

t: hat looks something like this

t: hat once I detect the mangled
Example

• Decode:  **errktreteoe.e**

• Hint:  • Is the last character, alphabetically first...
Burrows-Wheeler Transform

text -> bwt(text)

naive
create table of rotations sort table extract last row

text
aaccagg baccagg caggagg
caggagg daggagg
daggagg eaggagg
faggagg

text
aaccagg baccagg caggagg
caggagg daggagg
daggagg eaggagg
faggagg

bwt(text) -> text

inverse
follow LF-mapping on bwt(text)

LF - mapping
shows where the row will be after a rotation

Occ - naive

Occ - full
store letter count

Occ - checkpoints
store some letter counts after every 'n' letters

suffix array/tree
create suffix array/tree length of suffixes find letter left of suffix

text
aaccagg baccagg caggagg
caggagg daggagg
daggagg eaggagg
faggagg

bwt(text)
q c a a c c
e

'C[c]' is the first occurrence of letter 'c' in the F column
LF can be calculated with:
lf(lc) = (hc) + occ(c, 1)

where f is the position of 'c' in text

Occ(c, 1) or rank of letter 'c' at position '1' is the count of the letter 'c' upto position '1'.
The Occ function is the most important factor in the speed of LF mapping implementation.

FM-index uses bwt(text) for full text search

FM-index is a memory efficient full text index
it uses bwt(text) as an index
it uses LF mapping property to search in the index
it requires memory linear to the size of the text
if compression is used then its memory use can be sublinear to the size of the text

searching
proof of the update

count
because count points to the first suffix and not to the root

location
can be used with improving similarity to text

Egon Elbre
University of Tartu
github.com/egonelbre/fm-index
text -> bwt(text)

**naïve**

create table of rotations
\[
\begin{align*}
\text{a}
\text{c}
\text{g} & \text{a} \\
\text{c}
\text{a}
\text{c} & \text{g} \\
\text{a}
\text{c} & \text{g} \\
\text{a} & \text{c} \\
\text{c} & \text{g} \\
\text{g} & \text{a} \\
\end{align*}
\]

sort table
\[
\begin{align*}
\text{a}
\text{c} & \text{a} \\
\text{a} & \text{c} \\
\text{c} & \text{g} \\
\text{g} & \text{a} \\
\text{g} & \text{a} \\
\end{align*}
\]

extract last row
\[
\begin{align*}
g & c \\
c & a \\
a & a \\
a & c \\
\end{align*}
\]

bwt(text)
\[
g c a a a a c
\]

**suffix array/tree**

create suffix array/tree
\[
\begin{align*}
\text{a} & \\
\text{a} & \text{c} \\
\text{c} & \text{g} \\
\text{g} & \\
\end{align*}
\]

length of suffixes
\[
\begin{align*}
1 & \\
5 & \\
7 & \\
4 & \\
6 & \\
3 & \\
2 & \\
\end{align*}
\]

find letter left of suffix
\[
\begin{align*}
g & \\
c & \\
a & \\
a & \\
a & \\
\end{align*}
\]

bwt(text)
\[
g c a a a a c
\]
LF mapping

shows where the row will be after a rotation

'C[c]' is the first occurrence of letter 'c' in the F column

LF can be calculated with:

\[
\text{LF}(i, c) = C[c] + \text{Occ}(c, i)
\]

for transformation 'c = bwt[i]'
FM-index uses BWT(text) for full text search

FM-index is a memory efficient full text index

- It uses BWT(text) as an index
- It uses LF mapping property to search in the index
- It requires memory linear to the size of the text
- If compression is used then its memory use can be sublinear to the size of the text

For proof of top update:
X = q[i+1..j-1]
\[ c = q[j] \]
\[ \text{top} = \text{LF}(	ext{top}, c) \]
\[ \text{bot} = \text{LF}(	ext{bot}, c) \]
\[ \text{if}\ (\text{top} \geq \text{bot}) \]
\[ \text{return} \ \text{none} \]
\[ \text{return} [\text{top}..\bot] \]

For search:
\[ \text{top}, \bot = \text{search}(q) \]
\[ \text{return} \ \text{bot} - \text{top} \]