**Advanced Algorithmics (6EAP)**  
MTAT.03.238  
**Heaps**  
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2020 Fall

**Priority queue**

- Insert $Q, x$
- Retrieve $x$ from $Q$ s.t. $x.value$ is min (or max)
- Sorted linked list:
  - $O(n)$ to insert $x$ into right place
  - $O(1)$ access-min, $O(1)$ delete-min

**Binary heap**

- Complete — missing nodes only at the lowest level
- Heap property — on any path the parent has higher priority than child
- Typically: min-heaps
- Priority queue:
  - $\text{insert}(Q,x)$
  - $\text{pop}(Q)$

**Complete Binary Trees**

**Array Storage**

- Fill the array following a breadth-first traversal:

**Heap/Priority queue**

- Find min/Delete; Insert;
- Decrease key (change value of the key)
- Merge two heaps ...

**Binomial heaps**:

- **Performance**: All of the following operations work in $O(\log n)$ time on a binomial heap with $n$ elements:
  - Insert a new element to the heap
  - Find the element with minimum key
  - Delete the element with minimum key from the heap
  - Decrease key of a given element
  - Delete a given element from the heap
  - Merge two given heaps to one heap
  - Finding the element with minimum key can also be done in $O(1)$ by using an additional pointer to the minimum.

```
left(i) = i*2  
right(i) = i*2+1  
parent(i) =   i/2
```
Binomial heaps, Fibonacci heaps, and applications

http://www.cs.tau.ac.il/~dannyf/tao09/ht09s.htm
Dan Feldman

Some links

- [http://www.cse.yorku.ca/~aaw/Jason/FibonacciHeapAnimation.html](http://www.cse.yorku.ca/~aaw/Jason/FibonacciHeapAnimation.html)
- [http://www.jucs.org/jucs_7_5/animation_for_teaching_purposes/Lauer_T.html](http://www.jucs.org/jucs_7_5/animation_for_teaching_purposes/Lauer_T.html)

Binomial trees
Lemma 20.1

- For the binomial tree $B_k$,
  1. there are $2^k$ nodes,
  2. the height of the tree is $k$,
  3. there are exactly $\binom{k}{i}$ nodes at depth $i$ for $i = 0, 1, \ldots, k$, and
  4. the root has degree $k$, which is greater than that of any other node; moreover if the children of the root are numbered from left to right by $k - 1, k - 2, \ldots, 0$, child $i$ is the root of a subtree $B_i$.

Properties of binomial trees

1) $|B_k| = 2^k$
2) degree(root($B_k$)) = $k$
3) depth($B_k$) = $k$

$\Rightarrow$ The degree and depth of a binomial tree with at most $n$ nodes is at most $\log(n)$.

Define the rank of $B_k$ to be $k$.

Binomial heaps (def)

A collection of binomial trees with at most one of every rank. Items at the nodes, heap ordered.

Possible rep: Doubly link roots and children of every node. Parent pointers needed for delete.


Figure 20.4 The binomial tree $B_4$ with nodes labeled in binary by a postorder walk.
Binomial heaps (operations)

Operations are defined via a basic operation, called linking, of binomial trees: Produce a $B_k$ from two $B_{k-1}$, keep heap order.

Binomial heaps (ops cont.)

Basic operation is $\text{meld}(h1,h2)$:
Like addition of binary numbers.

\[
\begin{array}{cccc}
B_5 & B_4 & B_3 & B_1 \\
\hline \\
h1: & B_4 & B_3 & B_1 & B_0 \\
h2: & B_5 & B_4 & B_3 & B_0 \\
+ & & & \\
\hline \\
B_5 & B_4 & B_3 & B_0 \\
\end{array}
\]

The execution of BINOMIAL-HEAP-UNION.(a)
Binomial heaps $H_1$ and $H_2$.

Delete min

Find min (=1)
Extract tree
Split tree, reverse
Merge/meld
Decrease key (y=26 => y=7)

What is the time complexity?

```
# A.len = k
for i=1..n do Increment(A); # O(?)
```

```
Increment(A)
  1. i=0
  2. while i<A.len and A.i==1
  3. A[i] = 0
  4. i++
  5. if i < A.len
  6. A[i] = 1
```

Binomial heaps (ops cont.)

```
Findmin(h): obvious
Insert(x,h): meld a new heap with a single B_0 containing x, with h
deletemin(h): Chop off the minimal root. Meld the subtrees with h. Update minimum pointer if needed.
delete(x,h): Bubble up and continue like delete-min
decrease-key(x,h,i): Bubble up, update min ptr if needed
```

All operations take $O(\log n)$ time on the worst case, except
find-min(h) that takes $O(1)$ time.

Amortized analysis

We are interested in the worst case running time of a sequence of operations.

Example: binary counter

```
Increment(A)
  1. i=0
  2. while i<A.len and A.i==1
  3. A[i] = 0
  4. i++
  5. if i < A.len
  6. A[i] = 1
```

```
<table>
<thead>
<tr>
<th>Increment(A)</th>
<th>00000</th>
<th>00001</th>
<th>00010</th>
<th>00011</th>
<th>00100</th>
<th>00101</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

• In the amortized running time analysis we pretend that very fast operations take a little bit longer than they actually do.
• This additional time is then later subtracted from the actual running time of slow operations.
• The amount of time saved for later use is measured at any given moment by a potential function.

Incrementing binary counter

```
<table>
<thead>
<tr>
<th>value</th>
<th>bits 0</th>
<th>bits 1</th>
<th>Total cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0 0 0</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0 0 0 1</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>2.</td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>0 0 0 1</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>0 0 1 0</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>0 0 1 1</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>6.</td>
<td></td>
<td></td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>0 0 1 1</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>0 1 0 0</td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>9</td>
<td>0 1 0 0</td>
<td></td>
<td>18</td>
</tr>
<tr>
<td>10</td>
<td>0 1 0 1</td>
<td></td>
<td>19</td>
</tr>
<tr>
<td>11</td>
<td>0 1 0 0</td>
<td></td>
<td>22</td>
</tr>
<tr>
<td>12</td>
<td>0 1 1 1</td>
<td></td>
<td>23</td>
</tr>
<tr>
<td>13</td>
<td>0 1 1 1</td>
<td></td>
<td>25</td>
</tr>
<tr>
<td>14</td>
<td>0 1 1 1</td>
<td></td>
<td>26</td>
</tr>
<tr>
<td>15</td>
<td>1 0 0 0</td>
<td></td>
<td>31</td>
</tr>
<tr>
<td>16</td>
<td>1 0 0 0</td>
<td></td>
<td>32</td>
</tr>
</tbody>
</table>
```
Amortized analysis (Cont.)

On the worst case increment takes $O(k)$.

$k = \#\text{digits}$

What is the complexity of a sequence of increments (on the worst case) ?

Define a potential of the counter:

$\Phi (c) = ?$

Amortized(increment) = actual(increment) + $\Delta \Phi$

$\sum_{i} \text{Amortized(increment}_i) = \sum_{i} \text{actual(increment}_i) + \Phi_n - \Phi_0$

Amortized analysis (Cont.)

Amortized(increment) = actual(increment) + $\Phi_n - \Phi_0$

Amortized(increment) = actual(increment) + $\Phi_2 - \Phi_1$

$\sum_{i} \text{Amortized(increment}_i) = \sum_{i} \text{actual(increment}_i) + \Phi_n - \Phi_0$

Amortized analysis (Cont.)

Define a potential of the counter:

$\Phi (c) = \#\text{(ones)}$

Amortized(increment) = actual(increment) + $\Delta \Phi$

Amortized(increment) = $1 + \#(1 \Rightarrow 0) + 1 - \#(1 \Rightarrow 0) = O(1)$

$\Rightarrow$ Sequence of $n$ increments takes $O(n)$ time

Binomial heaps - amortized ana.

$\Phi (\text{collection of heaps}) = \#(\text{trees})$

Amortized cost of insert $O(1)$

Amortized cost of other operations still $O(\log n)$

Sizes: 1, 2, 4, 8, 16, 32 …

Binary integer representation as bitvector.

Binomial heaps + lazy meld

Allow more than one tree of each rank.

Meld (h1, h2):

• Concatenate the lists of binomial trees.

• Update the minimum pointer to be the smaller of the minimums

$O(1)$ worst case and amortized.
Binomial heaps + lazy meld

As long as we do not do a delete-min our heaps are just doubly linked lists:

```
  9  8  9  11  4  6
```

Delete-min : Chop off the minimum root, add its children to the list of trees.
Successive linking: Traverse the forest keep linking trees of the same rank, maintain a pointer to the minimum root.

Possible implementation of delete-min is using an array indexed by rank to keep at most one binomial tree of each rank that we already traversed.

Once we encounter a second tree of some rank we link them and keep linking until we do not have two trees of the same rank. We record the resulting tree in the array

```
Amortized(delete-min) =
= (#links + max-rank) - #links
= O(log(n))
```

Fibonacci heaps (Fredman & Tarjan 84)

Want to do decrease-key(x,h,d) faster than delete+insert.

Ideally in O(1) time.

Why?

Dijkstra’s shortest path algorithm

Let G = (V,E) be a weighted (weights are non-negative) undirected graph, let s \in V. Want to find the distance (length of the shortest path), d(s,v) from s to every other vertex.

Application #2 : Prim’s algorithm for MST

Start with T a singleton vertex.
Grow a tree by repeating the following step:
Add the minimum cost edge connecting a vertex in T to a vertex out of T.

Application #2 : Prim’s algorithm for MST

Maintain the vertices out of T but adjacent to T in a heap.
The key of a vertex v is the weight of the lightest edge (v,w) where w is in the tree.
Iteration: Do a delete-min. Let v be the minimum vertex and (v,w) the lightest edge as above. Add (v,w) to T. For each edge (w,u) where u \in T,
if key(u) = \infty insert u into the heap with key(u) = w(w,u)
if w(w,u) < key(u) decrease the key of u to be w(w,u).

With regular heaps O(m log(n)).
With F-heaps O(n log(n) + m).
Finding the minimum node

- The minimum node of a Fibonacci heap $H$ is given by the pointer $\text{min}[H]$, so we can find the minimum node in $O(1)$ actual time. Because the potential of $H$ does not change, the amortized cost of this operation is equal to its $O(1)$ actual cost.

Fibonacci heaps (cont.)

Decrease-key $(x,h,d)$: indeed cuts the subtree rooted by $x$ if necessary as we showed.

In addition we maintain a mark bit for every node. When we cut the subtree rooted by $x$ we check the mark bit of $p(x)$. If it is set then we cut $p(x)$ too. We continue this way until either we reach an unmarked node in which case we mark it, or we reach the root.

This mechanism is called cascading cuts.
Suggested implementation for decrease-key(x,h,δ):
If x with its new key is smaller than its parent, cut the subtree rooted at x and add it to the forest. Update the minimum pointer if necessary.

Decrease-key (cont.)

Does it work?

Obs1: Trees need not be binomial trees any more.
Do we need the trees to be binomial?
Where have we used it?
In the analysis of delete-min we used the fact that at most log(n) new trees are added to the forest. This was obvious since trees were binomial and contained at most n nodes.

Decrease-key (cont.)

Such trees are now legitimate.
So our analysis breaks down.
Fibonacci heaps (cont.)

We shall allow non-binomial trees, but will keep the degrees logarithmic in the number of nodes.

- Rank of a tree = degree of the root.
- Delete-min: do successive linking of trees of the same rank and update the minimum pointer as before.
- Insert and meld also work as before.

Fibonacci heaps (delete)

Delete(x,h): Cut the subtree rooted at x and then proceed with cascading cuts as for decrease key. Chop off x from being the root of its subtree and add the subtrees rooted by its children to the forest. If x is the minimum node do successive linking.

Fibonacci heaps (analysis)

- The potential of a Fibonacci heap is given by
  - Potential = \( t + 2m \) where \( t \) is the number of trees in the Fibonacci heap, and \( m \) is the number of marked nodes. A node is marked if at least one of its children was cut since this node was made a child of another node (all roots are unmarked).

Fibonacci heaps (analysis)

Lemma 1: Let x be any node in an F-heap. Arrange the children of x in the order they were linked to x, from earliest to latest. Then the i-th child of x has rank at least i-2.

Proof:
When the i-th node was linked it must have had at least i-1 children. Since then it could have lost at most one.
Fibonacci heaps (analysis)

Corollary 1: A node x of rank k in a F-heap has at least \( f^k \) descendants, where \( f = (1 + \sqrt{5})/2 \) is the golden ratio.

Proof:
Let \( s_k \) be the minimum number of descendants of a node of rank k in a F-heap.
By Lemma 1, \( s_k \geq \sum_{i=0}^{k} s_i + 2 \)
\[ s_0 = 1, \quad s_1 = 2 \]

Fibonacci heaps (analysis)

Proof (cont):
Fibonacci numbers satisfy
\[ F_{k+2} = \sum_{i=2}^{k} F_i + 2, \text{ for } k \geq 2, \quad F_2 = 1 \]
so by induction \( s_k \geq F_{k+2} \)
It is well known that \( F_{k+2} \geq f^k \)
It follows that the maximum degree \( k \) in a F-heap with \( n \) nodes is such that \( f^k \leq n \)
so \( k \leq \log(n) / \log(f) = 1.4404 \log(n) \)

---

**Make-Fibonacci-Heap**

\[ n[H] := 0 \]
\[ \text{return } H \]

**Fibonacci-Heap-Minimum**

\[ \text{return } \text{min}[H] \]

---

**CONSOLIDATE**(H)

for \( i = 0 \) to \( D(n[H]) \)
    if \( A[i] = \text{NIL} \)
        \( A[i] = \text{NIL} \)
        do each node \( w \) in the root list of \( H \)
            \( x := w \)
            \( d := \text{degree}(x) \)
            do \( y := A[d] \)
                if \( \text{key}(x) > \text{key}(y) \)
                    exchange \( x \rightleftharpoons y \)
                    Fibonaci-Heap-Link(H, y, x)
                    \( A[d] := \text{NIL} \)
                    \( d := d + 1 \)
                \( A[d] := x \)
        for \( i = 0 \) to \( D(n[H]) \)
            if \( A[i] = \text{NIL} \) or \( \text{key}(A[i]) > \text{key}(\text{min}(H)) \)
                then add \( A[i] \) to the root list of \( H \)
                if \( \text{min}(H) = \text{NIL} \) or \( \text{key}(A[i]) > \text{key}(\text{min}(H)) \)
                    then \( \text{min}(H) = A[i] \)

**Fibonacci-Heap-Link**(H, y, x)

remove y from the root list of H
make y a child of x
\( \text{degree}[x] := \text{degree}[x] + 1 \)
\( \text{mark}[y] := \text{FALSE} \)
Fibonacci-Heap-Union\((H1, H2)\)
\[
H := \text{Make-Fibonacci-Heap()}
\]
\[
\text{min}[H] := \text{min}[H1]
\]
Concatenate the root list of \(H2\) with the root list of \(H\)
if \((\text{min}[H1] = \text{NIL})\) or \((\text{min}[H2] < \text{NIL} \text{ and } \text{min}[H2] < \text{min}[H1])\)
then \(\text{min}[H] := \text{min}[H2]\)
\[
\text{n}[H] := \text{n}[H1] + \text{n}[H2]
\]
free the objects \(H1\) and \(H2\)
return \(H\)

Fibonacci-Heap-Insert\((H, x)\)
\[
\text{degree}[x] := 0
\]
\[
\text{p}[x] := \text{NIL}
\]
\[
\text{child}[x] := \text{NIL}
\]
\[
\text{left}[x] := x
\]
\[
\text{right}[x] := x
\]
\[
\text{mark}[x] := \text{FALSE}
\]
concatenate the root list containing \(x\) with root list \(H\)
if \(\text{min}[H] = \text{NIL}\) or \(\text{key}[x] < \text{key}[\text{min}[H]]\)
then \(\text{min}[H] := x\)
\[
\text{n}[H] := \text{n}[H] + 1
\]

Fibonacci-Heap-Extract-Min\((H)\)
\[
z := \text{min}[H]
\]
if \(x \neq \text{NIL}\)
then for each child \(x\) of \(z\)
\[
do \text{add } x \text{ to the root list of } H
\]
\[
\text{p}[x] := \text{NIL}
\]
\[
\text{remove } z \text{ from the root list of } H
\]
if \(z = \text{right}[z]\)
then \(\text{min}[H] := \text{NIL}\)
else \(\text{min}[H] := \text{right}[z]\)
CONSOLIDATE\((H)\)
\[
\text{n}[H] := \text{n}[H] - 1
\]
return \(z\)

Fibonacci-Heap-Decrease-Key\((H, x, k)\)
if \(k > \text{key}[x]\)
then error "new key is greater than current key"
\[
\text{key}[x] := k
\]
\[
y := \text{p}[x]
\]
if \(y = \text{NIL}\) and \(\text{key}[x] < \text{key}[y]\)
then \(\text{CUT}(H, x, y)\)
CASCADING-CUT\((H, y)\)
if \(\text{key}[x] < \text{key}[\text{min}[H]]\)
then \(\text{min}[H] := x\)

CUT\((H, x, y)\)
Remove \(x\) from the child list of \(y\), decrementing \(\text{degree}[y]\)
Add \(x\) to the root list of \(H\)
\[
\text{p}[x] := \text{NIL}
\]
\[
\text{mark}[x] := \text{FALSE}
\]
CASCADING-CUT\((H, y)\)
\[
z := \text{p}[y]
\]
if \(z = \text{NIL}\)
then if \(\text{mark}[y] = \text{FALSE}\)
then \(\text{mark}[y] := \text{TRUE}\)
else \(\text{CUT}(H, y, z)\)
CASCADING-CUT\((H, z)\)

Fibonacci-Heap-Delete\((H, x)\)
Fibonacci-Heap-Decrease-Key\((H, x, \text{-infinity})\)
Fibonacci-Heap-Extract-Min\((H)\)
van Emde Boas tree

A van Emde Boas tree (or van Emde Boas priority queue), also known as a vEB tree, is a tree data structure which implements an associative array with m-bit integer keys. It performs all operations in $O(\log m)$ time. Notice that $m$ is the size of the keys — therefore $O(\log m)$ is $O(\log \log n)$ in a full tree, exponentially better than a self-balancing binary search tree. They also have good space efficiency when they contain a large number of elements, as discussed below. They were invented by a team led by Peter van Emde Boas in 1977.

Wikipedia. Mar 20, 2010
Sketches of Algorithms

- Insert($i$)
- Delete($i$)
- Member($i$)
- Min
- Predecessor

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