Industrial Master’s Programme:

- for Software Engineering and Computer Science curricula
- 50% of the studies will take place in collaboration with a partner company
- 3 semesters (17 months)
- internship, practical course assignments, series of seminars designed specifically for Industrial Master students and master’s thesis
- scholarship starting from 660€/month
- deadline for applying: 29. September 2019
- more information: cs.ut.ee/en/studying/industrial-masters-programme

Contents

- Tree as a data model
- Data structures
- Search trees
  - binary trees and balancing
  - (2,4)-trees, B-trees
  - k-d trees
- Heaps
- Union-find problem
- ...

Trees

- Some of the very basic essence of computer science and programming

  - Chapter 5 – “The Tree Data Model” (pp 223-285) in
  - Foundations of Computer Science: C Edition
    - Alfred V. Aho, Jeffrey D. Ullman
    - W. H. Freeman (October 15, 1994)

Tree

- Acyclic graph
  - root of a tree
    - children, parents, siblings, internal nodes, leaves

  - Binary tree – node has 0, 1, or 2 children

Data model

- Abstraction

  - File directory system
  - Hierarchical organisation structure
    - divide and conquer
  - Hierarchical controlled vocabulary (simple ontology)
  - syntactic structure of a (sentence in a) language
  - syntax – e.g. paired parentheses
  - ...
Example: XHTML and CSS

• The nested tags define sub-trees

```html
<html>
<head>
<title>Hello World!</title>
</head>
<body>
<h1>This is a <u>Heading</u></h1>
<p>This is a paragraph with some <u>underlined</u> text.</p>
</body>
</html>
```

Example: XHTML and CSS

• The nested tags define sub-trees

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</body>
</html>
```

Example: XHTML and CSS

• This defines a single tree

```
root
 A
   |
  /|
 / |
B C D
|
E
```

Example: XHTML and CSS

• This may be rendered by a web browser

```
root
 A
   |
  /|
 / |
B C D
|
E
```

Terminology

- root: A
- leaf: {D,F,G,J,K,L,M}
- degree B = 2
- degree I = 3
- Every node has 1 parent except root has 0 parents
- Depth = 3
- Level C,E,I,M = 2
- width = 6 / at level 3/
- successor = children siblings (F,G), (J,K,L)
- Path to K = A,H,I,K

Example: XHTML and CSS

• List is also a tree:

```
root
 A
   |
  /|
 / |
B C
|
D
```
Terminology

Descendants (of B) = B,C,D,E,F,G
= a subtree with a root B

Ancestors of I = I,H,A

Every node is connected via a path to root.

Topologically equal to previous slide

Depends on application if order is important or not

Trie for P={he, she, his, hers}

Implementation of branching

Binary trees
Binary Trees
Definition: Any node can have 0, 1 or 2 children

- A **full node** is a node where both the left and right sub-trees are non-empty trees

Legend:
- full nodes
- neither
- leaf nodes

Basic node structure

**Binary Trees**

- An **empty node** or a **null sub-tree** is any location where a new leaf node could be inserted

**Perfect Binary Trees: Definition**

- A perfect binary tree of height $h$ is a binary tree where
  - All leaves have the same depth $h$
  - All other nodes are full

**Perfect Binary Trees Examples**

- Perfect binary trees of height $h = 0, 1, 2, 3$ and $4$
Perfect Binary Trees

\[ 2^h + 1 - 1 \text{ Nodes} \]

- Using the recursive definition, both sub-trees are perfect trees of height \( h - k - 1 \)
- By assumption, each sub-tree has \( 2^{k+1} - 1 \) nodes
- Therefore the total number of nodes is
  \[
  (2^k + 1) + 1 + (2^k + 1) = 2^k + 2 - 1
  \]

Complete Binary Trees: Definition

- A complete binary tree filled at each depth from left to right:

Complete Binary Trees: Array Storage

- Fill the array following a breadth-first traversal:
  
  \[
  \text{left}(i) = i \times 2 \\
  \text{right}(i) = i \times 2 + 1 \\
  \text{parent}(i) = \lfloor i / 2 \rfloor
  \]

Complete Binary Trees: Array Storage

- To insert another node while maintaining the complete-binary-tree structure, we must insert into the next array location

Traversal of a binary tree

function: Tree-Walk( x )
Traversal of a binary tree

Tree-Walk( x )
if x ≠ NULL then
    Tree-Walk( left(x) )
    Tree-Walk( right(x) )

Traversal of a binary tree - size

int Tree-Size( x )
if x == NULL then return 0
return
    Tree-Size( left(x) ) +
    Tree-Size( right(x) ) + 1

Traversal of a binary tree – parenthesisation

Tree-Walk( x )
if x ≠ NULL then
    // pre-order operations
    Tree-Walk( left(x) )
    // in-order operations
    Tree-Walk( right(x) )
    // post-order operations

Parenthesisation

( 3 ( 9 ( 14 (17) (15) ) (10 (13) (23) ) ) ) ...

Binary Trees
Application: Expression Trees

• Expression trees
  3(2a + c + a) + b/3 + (a − 2)
Binary Trees

Application: Expression Trees
- Internal nodes store operators
- Leaves store operands
- No node has just one sub tree
- The order is not relevant for addition and multiplication (commutative)
- The order is relevant for subtraction and division (non-commutative)
- To ignore order completely, represent subtraction and division as unary operators

\[(a/b) = a \cdot b^{-1} \quad (a - b) = a + (-b)\]

Evaluate the expression

```c
int Eval-Tree( x )
int val1, val2;
if x->op == 'l' return x->value; // x is a leaf, integer value
else
    val1 = Eval-Tree( x->left );
    val2 = Eval-Tree( x->right );
    switch ( n->op ) {
        case +: return val1 + val2;
        case -: return val1 - val2;
        case *: return val1 * val2;
        case /: return val1 / val2;
    }
```

General Trees: Design

- Children – in a linked list

Traversal of a general tree

```c
Tree-Walk( x )
if x ≠ NULL then
    foreach c in children(x)
        Tree-Walk( c )
```
Traversals of a general tree

Tree-Walk( x )
if x ≠ NULL then
  // pre-order operations
  foreach c in children(x)
    Tree-Walk( c )
  // post-order operations

Depth-first Traversal

- We note that each node could be visited twice in such a scheme
  - the first time the node is approached, and
  - the last time it is approached.

Pre-order Depth-first Traversal

- Visiting each node first results in the sequence
  A, B, C, D, E, F, G, H, I, J, K, L, M

Post-order Depth-first Traversal

- Visiting the nodes with their last visit:

Parenthesised tree serialisation

- Passing such a visitor results in the output:
  (A(B(C(D)(E(F)(G)))(H(I(J(K(L)(M)))))
Breadth-First Traversal

• Breadth-first traversal would visit the nodes in the order:

Breadth-First (x)

```c
enqueue(Q, x)
while not empty(Q)
  x = dequeue(Q)
  print x->name  // process node x
  foreach c in next-child(x)
    enqueue(Q, c)
```

Printing Directories

• Given the directory structure

```
/  
usr/  
  bin/  
  local/  
var/  
  adm/  
  cron/  
  log/  
```

Exercise

• Print the following statistics for a given (e.g. current working) directory:
  – subdirectory size (# of all subdirectories and files)
  – depth (maximal height)
  – width at all levels of depth...
  – maximal depth
  – largest directory in nr of subdirs and files in that directory
  – ...

Binary Search Tree (BST)

• Data structure for (Dynamic) Dictionary

• MIT

• Binary tree where values of the keys have a special order:
  values(left subtree) < value(root) <= values(right subtree)
50% did correct BST. Others – empty or mistakes.

2, 5, 6, 1, 3 into BST

<table>
<thead>
<tr>
<th>What is the time complexity of</th>
<th>Worst case</th>
<th>Average case</th>
<th>Could you implement it without consulting literature?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Quicksort</td>
<td>1.</td>
<td>1.</td>
<td>1.</td>
</tr>
<tr>
<td>2. Merge Sort</td>
<td>2.</td>
<td>2.</td>
<td>2.</td>
</tr>
<tr>
<td>3. Heap Sort</td>
<td>3.</td>
<td>3.</td>
<td>3.</td>
</tr>
<tr>
<td>4. Radix sort</td>
<td>4.</td>
<td>4.</td>
<td>4.</td>
</tr>
<tr>
<td>5. Theoretically best</td>
<td>5.</td>
<td>5.</td>
<td>5.</td>
</tr>
<tr>
<td>comparison based sort</td>
<td>6.</td>
<td>6.</td>
<td>6.</td>
</tr>
<tr>
<td>6. Theoretically best</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sorting (any)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Draw a binary search tree (BST) after inserting: 2, 5, 6, 1, 3 into an initially empty tree

What is the purpose of BST?

Examples

- Here we see a complete binary search tree, and a binary search tree which is close to being complete -- balanced

Examples

- There are many different representations of the same ordered data:

**Operations on dynamic sets**

SEARCHES, II
A query that, given a set $S$ and a key value $k$, returns a pointer to an element in $S$ such that $key[x] = k$, or NIL if such an element belongs to $S$.

**INSERT**, $x$
A modifying operation that augments the set $S$ with the element pointed to by $x$. We usually assume that any fields in element $x$ needed by the set implementation have already been initialized.

**DELETE**, $x$
A modifying operation that, given a pointer $x$ to an element in the set $S$, removes $x$ from $S$. (Note that the operation uses a pointer to an element $x$, not a key value.)

**MINIMUM**, $S$
A query on a totally ordered set $S$ that returns a pointer to the element of $S$ with the smallest key.

**MAXIMUM**, $S$
A query on a totally ordered set $S$ that returns a pointer to the element of $S$ with the largest key.

**SUCCESSOR**, $S, x$
A query that, given an element $x$ whose key is from a totally ordered set $S$, returns a pointer to the next larger element in $S$, or NIL if $x$ is the maximum element.

**PREDECESSOR**, $S, x$
A query that, given an element $x$ whose key is from a totally ordered set $S$, returns a pointer to the next smaller element in $S$, or NIL if $x$ is the minimum element.

**Operations - search**

TREE-SEARCH $(x, k)$
1. if $x = \text{NIL}$ or $k = x.key$
2. then return $x$
3. if $k < x.key$
4. then return TREE-SEARCH$(x.left, k)$
5. else return TREE-SEARCH$(x.right, k)$
Iterative search

ITERATIVE-TREE-SEARCH (x, k)
1 while x ≠ NIL and k ≠ x.key
2 if k < x.key
3 then x = x.left
4 else x = x.right
5 return x

(Tail) Recursion "unrolling" – should be more efficient

Min and Max

Tree-Minimum (x)
1 while left[x] ≠ NIL
2 x = left[x]
3 return x

Tree-Maximum (x)
1 while right[x] ≠ NIL
2 x = right[x]
3 return x

Successor

Tree-Successor (x)
1 if right[x] ≠ NIL
2 then return Tree-Minimum(right[x])
3 y = parent[x]
4 while y ≠ NIL and x = right[y]
5 x = y; y = parent[y]
6 return y

Insert a node

• Find such a node where “next” position is missing...

Remove

• Suppose we wish to remove a node
• There are three situations: the node being removed
  – is a leaf node,
  – has exactly one child, or
  – is a full node (two children).

Remove

• If it is a leaf node, we can remove it:
Remove

• If the node has only one child, we can promote that child (with all the subtree underneath):

Example

• Consider the following tree
  • We will twice remove the root

Example

• First, to remove 15, it is a full node
  • We find the minimum element in the right sub-tree

Example

• We promote 42 to the root
  • Proceed to remove 42 from the right sub-tree

Example

• This has one child, so we promote the entire sub-tree to replace 42

Remove

• If it is a full node, we copy the minimum element from the right sub-tree
• Recursively delete the value we copied
Example

• The root has been deleted, and the result is still a binary search tree

Example

• Next, let us remove 42
• Once again, it is a full node, so get the minimum element in the right sub-tree

Example

• We promote 45 to the root and proceed to delete 45 from the right sub-tree

Example

• The node 45 is a leaf node, so we may simply remove it

Example

• Thus, the final tree, having removed 15 and then 42 is

Reading

• CLRS: Binary Search Trees

• Visualisations:
Complexity...

- (Almost) all operations depend on the depth of the tree (or node affected)

- Binary search tree can get unbalanced, depth $O(n)$

- How to ensure this does not happen?

Balanced Binary Search Trees

- MIT [Link]

- AVL trees
- 2-3 trees
- 2-3-4 trees
- B-trees
- Red-black trees

Balance

- If elements are added in random, tree is “automatically balanced” on average
- Otherwise: we must re-balance it ourselves...

AVL-trees

- Adelson-Velskii and Landis
- [Link]

- In an AVL tree, the heights of the two child subtrees of any node differ by at most one;

- The AVL tree is named after its two inventors, G.M. Adelson-Velsky and E.M. Landis, who published it in their 1962 paper "An algorithm for the organization of information."

- In an AVL tree, the heights of the two child subtrees of any node differ by at most one;

- Difference: -1, 0, 1

- Re-balance using rotations when getting out of balance...

- $O(\log n)$ normal operations

- Up to $O(\log n)$ re-balancing operations of $O(1)$

- an AVL tree's height is limited to $1.44 \log n$
Height of an AVL Tree

• If \( n = 88 \), the worst- and best-case scenarios differ in height by only 2:

\[
\log_2(10^6) - 1 \approx 19
\]

\[
\log_2(10^6 / 1.8944) \approx 28
\]

An AVL tree’s height is strictly less than:

\[
\log_2(\sqrt{n+1}) - 2 = \frac{\log(\sqrt{n+1})}{\log(2)} - 2 = \log(\sqrt{n+1}) - 2 = 1.144 \log(n+2) - 8.328
\]

where \( \varphi \) is the golden ratio.

Red-Black trees

1. A node is either red or black.
2. The root is black. (This rule is used in some definitions and not others. Since the root can always be changed from red to black but not necessarily vice-versa this rule has little effect on analysis.)
3. All leaves are black.
4. Both children of every red node are black.
5. Every simple path from a node to a descendant leaf contains the same number of black nodes.

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Red-black trees

This data structure requires an extra one-bit color field in each node.

Red-black properties:

1. Every node is either red or black.
2. The root and leaves (NIL’s) are black.
3. If a node is red, then its parent is black.
4. All simple paths from any node \( x \) to a descendant leaf have the same number of black nodes = black-height(\( x \)).
Example of a red-black tree

Height of a red-black tree

**Theorem.** A red-black tree with \( n \) keys has height \( h \leq 2 \lg(n + 1) \).

**Proof.** (The book uses induction. Read carefully.)

**Intuition:**
- Merge red nodes into their black parents.
- This process produces a tree in which each node has 2, 3, or 4 children.
- The 2-3-4 tree has uniform depth \( h' \) of leaves.

Proof (continued)

**Corollary.** The queries SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR all run in \( O(\lg n) \) time on a red-black tree with \( n \) nodes.

Query operations
Modifying operations

The operations \texttt{INSERT} and \texttt{DELETE} cause modifications to the red-black tree:

- the operation itself,
- color changes,
- restructuring the links of the tree via "rotations".

Rotations

Rotations maintain the inorder ordering of keys:

\[ a \preceq \beta, b \preceq \gamma \Rightarrow a \preceq \beta \preceq b \preceq \gamma \preceq c. \]

A rotation can be performed in \( O(1) \) time.

Insertion into a red-black tree

\textbf{IDEA:} Insert \( x \) in tree. Color \( x \) red. Only red-black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

\textbf{Example:}

```
           2
          / \  \\
         10  12
        /   /  \\
       11  20  26
```

Insertion into a red-black tree

\textbf{IDEA:} Insert \( x \) in tree. Color \( x \) red. Only red-black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

\textbf{Example:}

- Insert \( x = 15 \).
- Recolor, moving the violation up the tree.

```
           2
          / \  \\
         10  12
        /   /  \\
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```

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- Insert \( x = 15 \).
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- \texttt{RIGHT-ROTATE(18)}.

```
           2
          / \  \\
         10  12
        /   /  \\
       11  20  26
```

Insertion into a red-black tree

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\textbf{Example:}

- Insert \( x = 15 \).
- Recolor, moving the violation up the tree.
- \texttt{RIGHT-ROTATE(18)}.
- \texttt{LEFT-ROTATE(18)} and recolor.

```
           2
          / \  \\
         10  12
        /   /  \\
       11  20  26
```

```
           2
          / \  \\
         10  12
        /   /  \\
       11  20  26
```
**Insertion into a red-black tree**

**IDEA:** Insert \( x \) in tree. Color \( x \) red. Only red-black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

**Example:**
- Insert \( x = 15 \).
- Recolor, moving the violation up the tree.
- **RIGHT-ROTATE(18).**
- **LEFT-ROTATE(7)** and recolor.

---

**Pseudocode**

\[
\text{RB-INSERT}(T, x) \\
\text{TREE-INSERT}(T, x) \\
\text{color}[x] \leftarrow \text{RED} \quad \Rightarrow \text{only RB property 3 can be violated} \\
\text{while} \ x \neq \text{root}(T) \text{ and color}[\text{p}(x)] = \text{RED} \\
\text{do if } [x] = \text{left}[p[x]] \\
\text{then } y \leftarrow \text{right}[p[x]] \quad \Rightarrow y = \text{aunt/uncle of } x \\
\text{if color}[y] = \text{RED} \quad \text{(Case 1)} \\
\text{else if } x = \text{right}[y] \quad \text{then (Case 2)} \quad \Rightarrow \text{Case 2 falls into Case 3} \\
\text{else ("then" clause with "left" and "right" swapped)} \\
\text{color[root(T)]} \leftarrow \text{BLACK}
\]

---

**Graphical notation**

Let \( A \) denote a subtree with a black root.

All \( A \)'s have the same black-height.

---

**Case 1**

![Diagram for Case 1](image)

Recolor

Push \( C \)'s black onto \( A \) and \( D \), and recurse, since \( C \)'s parent may be red.

---

**Case 2**

![Diagram for Case 2](image)

**LEFT-ROTATE(A)**

Transform to Case 3.

---

**Case 3**

![Diagram for Case 3](image)

**RIGHT-ROTATE(C)**

Done! No more violations of RB property 3 are possible.
Analysis

- Go up the tree performing Case 1, which only recolors nodes.
- If Case 2 or Case 3 occurs, perform 1 or 2 rotations, and terminate.

**Running time:** $O(\log n)$ with $O(1)$ rotations.

**RB-DELETE** — same asymptotic running time and number of rotations as RB-INSERT (see textbook).

Other ideas

- Balancing can be an independent process — at night?
- Many search&insert&delete processes, and few rebalancing processes
- Local locking. Must ensure no deadlocks occur!

Properties of Red-Black trees

- No overhead for searching — efficient
- 100-200 lines of code, many symmetric cases
- Left-Leaning Red-Black trees (LLRB)
  - Robert Sedgewick:

Splay trees (Sleator, Tarjan 1985)

- Self-adjusting BST
- Recently accessed elements are brought to the top of the tree
- **Repeated accesses will be executed faster!**
- No extra bookkeeping
- Average $\log(n)$ but worst case $O(n)$

2-3, 2-3-4, B-trees

- Binary trees are useful for memory-based data structures
- Large databases and disk based systems would benefit of fewer reads of larger block sizes
- Organise data in a search tree that minimizes disk accesses
A B-tree of order $m$ (the maximum number of children for each node) is a tree which satisfies the following properties:

- Every node has at most $m$ children.
- Every node (except root and leaves) has at least $m/2$ children.
- The root has at least two children if it is not a leaf node.
- All leaves appear in the same level, and carry information.
- A non-leaf node with $k$ children contains $k−1$ keys.

Half-full property ensures that ...

- two half-full nodes can be joined to make a legal node, and one full node can be split into two legal nodes (if there is room to push one element up into the parent).

Example: Two-level Insertion

- Inserting 29
- Leaf node is full, so we split it into two
Example: Two-level Insertion
• Parent node is full, so we must split it

Example: Two-level Insertion
• The root node must be updated

Example: Root Insertion
• Insert 67
• Leaf is full, so split it into two

Example: Root Insertion
• Parent is full, so split it into two

Example: Root Insertion
• Root is full, so split it into two

Example: Root Insertion
• Create a new root node
The creators of the B-tree structure, Rudolf Bayer and Ed McCreight, have not explained what, if anything, the B stands for. Douglas Comer suggests a number of possibilities:

- "Balanced," "Broad," or "Bushy" might apply [since all leaves are at the same level]. Others suggest that the "B" stands for Boeing [since the authors worked at Boeing Scientific Research Labs in 1972]. Because of his contributions, however, it seems appropriate to think of B-trees as "Bayer"-trees.

Variants of B-trees

- Keys and data in leaves or internal nodes
- Order statistics
- ...

Analogy between R-B and B-trees

- Multi-dimensional data
  - 2-dim \((x,y)\)
  - 3D \((x,\bar{x},z)\)
  - d-dim \((x_1,...,x_d)\)
- Does a point belong to a set?
- What is the closest point? (other data structures)
- ...

2-d data \((xy, gps, coordinates)\)

\[(7,2) (5,4) (2,3) (4,7) (9,6) (8,1) \ldots\]

k-d tree

- Multi-dimensional data
  - 2-dim \((x,y)\)
  - 3-D \((x_1, \bar{x}_1, z)\)
  - d-dim \((x_1, ..., x_d)\)
- Does a point belong to a set?
- What is the closest point? (other data structures)
- ...

2-D data \((x, y \text{ coordinates})\)

\[\text{2-D tree (x,y coordinates)}\]
Suppose we wish to partition the following points in a 2-dimensional $kd$-tree:

\[(0.01, 0.48), (0.02, 0.91), (0.02, 0.97), (0.03, 0.90), (0.04, 0.06), (0.04, 0.41), (0.04, 0.61), (0.05, 0.07), (0.05, 0.88), (0.05, 0.97), (0.06, 0.05), (0.06, 0.28), (0.08, 0.89), (0.09, 0.55), (0.23, 0.23), (0.29, 0.15), (0.33, 0.07), (0.37, 0.04), (0.41, 0.89), (0.46, 0.58), (0.54, 0.65), (0.55, 0.02), (0.55, 0.54), (0.56, 0.78), (0.68, 0.42), (0.73, 0.69), (0.74, 0.97), (0.94, 0.02), (0.95, 0.07), (0.97, 0.09), (0.97, 0.18)\]

The first step is to order the points based on the 1st coordinate and find the median:

\[(0.01, 0.48), (0.02, 0.91), (0.02, 0.97), (0.03, 0.90), (0.04, 0.06), (0.04, 0.41), (0.04, 0.61), (0.05, 0.07), (0.05, 0.88), (0.05, 0.97), (0.06, 0.05), (0.06, 0.28), (0.08, 0.89), (0.09, 0.55), (0.23, 0.23), (0.29, 0.15), (0.33, 0.07), (0.37, 0.04), (0.41, 0.89), (0.46, 0.58), (0.54, 0.65), (0.55, 0.02), (0.55, 0.54), (0.56, 0.78), (0.68, 0.42), (0.73, 0.69), (0.74, 0.97), (0.94, 0.02), (0.95, 0.07), (0.97, 0.09), (0.97, 0.18)\]

The median point, (0.29, 0.15), forms the root of our $kd$-tree.

This partitions the remaining points into two sets:

\[(0.04, 0.06), (0.04, 0.41), (0.04, 0.61), (0.05, 0.88), (0.05, 0.97), (0.06, 0.05), (0.06, 0.28), (0.08, 0.89), (0.09, 0.55), (0.23, 0.23), (0.29, 0.15), (0.33, 0.07), (0.37, 0.04), (0.41, 0.89), (0.46, 0.58), (0.54, 0.65), (0.55, 0.02), (0.55, 0.54), (0.56, 0.78), (0.68, 0.42), (0.73, 0.69), (0.74, 0.97), (0.94, 0.02), (0.95, 0.07), (0.97, 0.09), (0.97, 0.18)\]
**kd-Trees**

- This point creates the left child of the root

- Starting with the second partition, we also order these according to the 2nd coordinate:

  - (0.55, 0.02), (0.94, 0.02), (0.37, 0.04), (0.33, 0.07), (0.95, 0.07), (0.97, 0.09), (0.97, 0.18), (0.68, 0.42), (0.55, 0.54), (0.46, 0.58), (0.54, 0.65), (0.73, 0.69), (0.56, 0.78), (0.41, 0.89), (0.74, 0.97)

**kd-Trees**

- This point creates the right child of the root

- Next, ordering the partitioned elements by the 1st coordinate, we choose the medians to find the children of the left child (0.09, 0.55):

  - (0.01, 0.48), (0.04, 0.06), (0.04, 0.41), (0.05, 0.07), (0.06, 0.05), (0.06, 0.28), (0.23, 0.23), (0.02, 0.91), (0.02, 0.97), (0.03, 0.90), (0.04, 0.61), (0.05, 0.88), (0.08, 0.89), (0.05, 0.97), (0.08, 0.89)

**kd-Trees**

- Doing the same with the two right partitions, we get the children of the right child of the root:

  - (0.33, 0.07), (0.37, 0.04), (0.55, 0.02), (0.94, 0.02), (0.95, 0.07), (0.97, 0.09), (0.97, 0.18), (0.46, 0.58), (0.54, 0.65), (0.41, 0.89)

**kd-Trees**

- At the next level, we order the points again based on the 2nd coordinate and choose the medians:

  - (0.01, 0.48), (0.04, 0.06), (0.04, 0.41), (0.05, 0.07), (0.06, 0.05), (0.06, 0.28), (0.23, 0.23), (0.02, 0.91), (0.02, 0.97), (0.03, 0.90), (0.04, 0.61), (0.05, 0.88), (0.08, 0.89), (0.05, 0.97), (0.08, 0.89)
**kd-Trees**

- The result is a 2-dimensional $kd$-tree of the given 31 points

**kd-Trees**

- Finally, the last point, a leaf node, falls within the given box

**kd-Trees**

- A useful application of a $kd$-tree provides an efficient data structure for counting the number of points which fall within a given $k$-dimensional rectangle

**kd-Trees**

- This is used in image processing: locating objects within a scene, ray tracing, etc.
  - Find the points which lie in the quadrant $[0.5, 1] \times [0, 0.5]$

**kd-Trees**

- The traversal rules we will follow are:
  - we always match the coordinate corresponding to the level we are current at
  - if that coordinate is less than the corresponding interval of the box, we only need to visit the right sub-tree
  - if that coordinate is greater than the corresponding interval, we need only visit the left sub-tree
  - otherwise, we check if the root is in the box and we visit both sub-trees

**kd-Trees**

- Starting with the left sub-tree:
  - $0.94 \in [0.5, 1]$
  - We note that $(0.94, 0.02) \in [0.5, 1] \times [0, 0.5]$ and we visit both sub-trees
Nearest neighbour search

- kd-trees are not suitable for efficiently finding the nearest neighbour in high dimensional spaces.
- As a general rule, if the dimensionality is $D$, then number of points in the data, $N$, should be $N >> 2^D$.
- Otherwise, when $kd$-trees are used with high-dimensional data, most of the points in the tree will be evaluated and the efficiency is no better than exhaustive search.
- The problem of finding NN in high-dimensional data is thought to be $NP$-hard [2], and approximate nearest-neighbour methods are used instead.

High dimensionality

- Data often comes in high-dimensional form
- Curse of dimensionality
  - K-d tree nodes become empty after a few levels already...
  - Everything is “far” from everything else
  - Difference along even one dimension makes them far from each other
Random projection tree (construction)

Project all points on this vector.

Choose a split point (median of projected points in this case).

This creates a partition of the original data into two subsets.

Again choose a random vector.

NB! We must store a random vector and the median at each node of the tree.
Project all points on this new vector.

Define a split.

Repeat recursively until limit of points in each leaf is reached (3 in this case)
We need to find a query point. At each level of the tree we will project the query point on a random vector and figure from which side of the median this projection lies.

Leaf 1
By constructing several trees you can improve the accuracy of the algorithm.

<table>
<thead>
<tr>
<th>Methods average complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Building Time</strong></td>
</tr>
<tr>
<td>K-d Tree</td>
</tr>
<tr>
<td>RP Tree</td>
</tr>
</tbody>
</table>

Random projection tree

- Places hyper planes between two arbitrary points each time dividing hyperspace into two equal parts
- Also keeps track of number of points in the node

See also (Wikipedia)

- implicit kd-tree
- min/max kd-tree
- Quadtree
- Octree
- Bounding Interval Hierarchy
- Nearest neighbor search
- Klee’s measure problem
- kd-trie
Quadtree – divide area recursively to quadrants

Octree – 3D, 8 children

R-Tree

Quadtree

http://nns.tume-maailm.pri.ee/

• 2-dimensional
• 4 quadrants
• Either point or area based

R-Tree

Overlapping
Minimal bounding boxes
B-Tree analog on k-D
Variants

- R, R+, R*

**Difference between R+ trees and R trees**
R+ trees are a compromise between R trees and KD trees; they avoid overlapping of internal nodes by inserting an object into multiple leaves if necessary.

**R+ trees differ from R trees in that:**
- Nodes are not guaranteed to be at least half filled
- The entries of any internal node do not overlap
- An object ID may be stored in more than one leaf node

**Advantages**
Because nodes are not overlapped with each other, point query performance benefits since all spatial regions are covered by at most one node. A single path is followed and fewer nodes are visited than with the R-tree.

---

Priority queue

- Insert Q, x

- Retrieve next x from Q s.t. x.value is largest

- Sorted list implementation:
  - O(n) to insert x into right place
  - O(1) access, O(1) delete

---

Binary heap

**Complete — missing nodes only at the lowest level**

**Heap property — on any path parent has higher priority**

**Typically: min-heaps**

**Priority queue**

- insert (Q, x)
- pop Q

---

**Binary heap - Insert**

Insert into a next allowed place

Make sure heap property is restored
Binary heap – Insert – “Bubble up”

Use Array based implementation

Insert

```
insert( int A[], int x, int *last) {
(*last)++ ;
A[*last] = x;
bubbleUp( A, *last ) ;
}
```

Bubble up

```
BubbleUp( int A[], int i) {
while ( ( i>1 ) && A[i] > A[i/2] ){
    swap( A, i, i/2 ) ;
i=i/2;
}
}
```

void swap( int A[], int i, int j ) {
    int temp;
temp=A[i];
A[i]=A[j];
A[j]=temp;
}

Delete (max)

- Remove top value (make free space)
- Remove last element
- Insert to top value location, then bubble down to the correct place
**Binary heap – Delete – “Bubble down”**

Cost
- Insert – \(O(\log n)\)
- Delete – \(O(\log n)\)

**Heap-sort**

- Heapify the array
- while not empty
  - pop_largest
  - copy to next free place

**Heapify... in linear time**

- last \(n/2\) – ignore
- \(n/4\) - bubble down (at most by 1 level)
- \(n/8\) – bubble down (at most by 2 levels)
- ...

\[
\sum_{i=1}^{\log_2 n} \frac{in}{2^{i+1}} \leq \frac{n}{2} \sum_{i=1}^{\infty} \frac{in}{2^i}
\]

The number in the circle indicates the maximum times of swapping required when adding the node to the heap.
1/2 + 1/4 + 1/8 + 1/16 + ... = 1
+ 1/4 + 1/8 + 1/16 + ... = 1/2
1/8 + 1/16 + ... = 1/4
1/8 nodes, 3 steps down... Sum = 2

Introduction to Algorithms
6.046J/18.401J

Lecture 11
Augmenting Data Structures
• Dynamic order statistics
• Methodology
• Interval trees

Prof. Charles E. Leiserson

Dynamic order statistics

OS-SELECT(i, S): returns the i-th smallest element in the dynamic set S.
OS-RANK(x, S): returns the rank of x ∈ S in the sorted order of S’s elements.

IDEA: Use a red-black tree for the set S, but keep subtree sizes in the nodes.

Notation for nodes:

Example of an OS-tree

size[x] = size[left[x]] + size[right[x]] + 1

Selection

Implementation trick: Use a sentinel (dummy record) for NIL such that size[NIL] = 0.

OS-SELECT(x, i) returns i-th smallest element in the subtree rooted at x
k ← size[left[x]] + 1 k = rank(x)
if i = k then return x
if i < k
then return OS-SELECT(left[x], i)
else return OS-SELECT(right[x], i − k)
(OS-RANK is in the textbook.)

Example

OS-SELECT(root, 5)

Running time = O(h) = O(lg n) for red-black trees.
In computer science, an interval tree, also called a segment tree or segtree, is an ordered tree data structure to hold intervals. Specifically, it allows one to efficiently find all intervals that overlap with any given interval or point. It is often used for windowing queries, for example, to find all roads on a computerized map inside a rectangular viewport, or to find all visible elements inside a three-dimensional scene.

The trivial solution is to visit each interval and test whether it intersects the given point or interval, which requires $\Theta(n)$ time, where $n$ is the number of intervals in the collection. Since a query may return all intervals, for example if the query is a large interval intersecting all intervals in the collection, this is asymptotically optimal; however, we can do better by considering output-sensitive algorithms, where the runtime is expressed in terms of $m$, the number of intervals produced by the query.
**Following the methodology**

1. Choose an underlying data structure.
   - Red-black tree keyed on low (left) endpoint.
2. Determine additional information to be stored in the data structure.
   - Store in each node $x$ the largest value $m[x]$ in the subtree rooted at $x$, as well as the interval $int[x]$ corresponding to the key.

```
interval, key = interval.left
```

**Example interval tree**

```
17,19
  /   \
5,11  23
  /  \
4,8  15,18
     /  \n8    18
     /  \
7,10  10
```

$m[x] = \max \{ \text{high}[int[x]], m[\text{left}[x]], m[\text{right}[x]] \}$

**Modifying operations**

3. Verify that this information can be maintained for modifying operations.
   - **INSERT**: Fix $m$’s on the way down.
   - **Rotations**: Fixup = $O(1)$ time per rotation.

```
6.20  30  14
  \   /   /
  \  /  / \\
30  19 30
  /   \
11,15 19
```

Total INSERT time = $O(\log n)$; DELETE similar.

**New operations**

4. Develop new dynamic-set operations that use the information.

```
INTERVAL-SEARCH(i)
  x ← root
  while x ≠ NIL and (low[i] > high[m[x]])
    or (low[m[x]] > high[i])
    do i and $\text{int}[x]$ don’t overlap
    if $\text{left}[x] = \text{NIL}$ and $\text{low}[i] \leq m[\text{left}[x]]$
      then $x ← \text{left}[x]$
    else $x ← \text{right}[x]$
  return $x$
```

**Example 1: INTERVAL-SEARCH([14,16])**

```
5,11
  /   \
4,8  15,18
     /  \n8    18
     /  \
7,10  10
```

$x ← \text{root}$

$[14,16]$ and $[17,19]$ don’t overlap

$14 ≤ 18 ⇒ x ← \text{left}[x]$
**Example 1:** \( \text{INTERVAL-SEARCH}([14,16]) \)

- \( x \)
- \( 17.19 \)
- \( 23 \)
- \( 5.11 \)
- \( 18 \)
- \( 4.8 \)
- \( 8 \)
- \( 2.10 \)
- \( 10 \)

\([14,16]\) and \([5.11]\) don’t overlap

\( 14 > 8 \Rightarrow x \leftarrow \text{right}(x) \)

**Example 1:** \( \text{INTERVAL-SEARCH}([14,16]) \)

- \( x \)
- \( 17.19 \)
- \( 23 \)
- \( 5.11 \)
- \( 18 \)
- \( 4.8 \)
- \( 8 \)
- \( 7.10 \)
- \( 10 \)

\([14,16]\) and \([15,18]\) overlap

return \([15,18]\)

**Example 2:** \( \text{INTERVAL-SEARCH}([12,14]) \)

- \( x \)
- \( 17.19 \)
- \( 23 \)
- \( 5.11 \)
- \( 18 \)
- \( 4.8 \)
- \( 8 \)
- \( 7.10 \)
- \( 10 \)

\( x \leftarrow \text{root} \)

\([12,14]\) and \([17,19]\) don’t overlap

\( 12 \leq 18 \Rightarrow x \leftarrow \text{left}(x) \)

**Example 2:** \( \text{INTERVAL-SEARCH}([12,14]) \)

- \( x \)
- \( 17.19 \)
- \( 23 \)
- \( 5.11 \)
- \( 18 \)
- \( 4.8 \)
- \( 8 \)
- \( 7.10 \)
- \( 10 \)

\([12,14]\) and \([5.11]\) don’t overlap

\( 12 > 8 \Rightarrow x \leftarrow \text{right}(x) \)

**Example 2:** \( \text{INTERVAL-SEARCH}([12,14]) \)

- \( 17.19 \)
- \( 23 \)
- \( 5.11 \)
- \( 18 \)
- \( 4.8 \)
- \( 8 \)
- \( 7.10 \)
- \( 10 \)

\([12,14]\) and \([15,18]\) don’t overlap

\( 12 > 10 \Rightarrow x \leftarrow \text{right}(x) \)

**Example 2:** \( \text{INTERVAL-SEARCH}([12,14]) \)

- \( 17.19 \)
- \( 23 \)
- \( 5.11 \)
- \( 18 \)
- \( 4.8 \)
- \( 8 \)
- \( 7.10 \)
- \( 10 \)

\( x = \text{NIL} \Rightarrow \text{no interval that overlaps} \ [12,14]\ \exists \)
Analysis

Time = \( O(h) = O(\lg n) \), since INTERVAL-SEARCH does constant work at each level as it follows a simple path down the tree.

List all overlapping intervals:
- Search, list, delete, repeat.
- Insert them all again at the end.
Time = \( O(k \lg n) \), where \( k \) is the total number of overlapping intervals.
This is an output-sensitive bound.
Best algorithm to date: \( O(k + \lg n) \).

Correctness

**Theorem.** Let \( L \) be the set of intervals in the left subtree of node \( x \), and let \( R \) be the set of intervals in \( x \)'s right subtree.
- If the search goes right, then \( \{ \, i' \in L : i' \text{ overlaps } i \, \} = \emptyset. \)
- If the search goes left, then \( \{ \, i' \in L : i' \text{ overlaps } i \, \} = \emptyset \implies \{ \, i' \in R : i' \text{ overlaps } i \, \} = \emptyset. \)

In other words, it’s always safe to take only 1 of the 2 children: we’ll either find something, or nothing was to be found.

Correctness proof

**Proof.** Suppose first that the search goes right.
- If \( \text{left}[x] = \text{NIL} \), then we’re done, since \( L = \emptyset. \)
- Otherwise, the code dictates that we must have \( \text{low}[i] > m[\text{left}[x]]. \) The value \( m[\text{left}[x]] \) corresponds to the high endpoint of some interval \( j \in L \), and no other interval in \( L \) can have a larger high endpoint than \( \text{high}[j]. \)
  \[
  \text{high}[j] = m[\text{left}[x]] \\
  \text{low}(i)
  \]
- Therefore, \( \{ \, i' \in L : i' \text{ overlaps } i \, \} = \emptyset. \)
Proof (continued)

Suppose that the search goes left, and assume that \( \{j' \in L: i' \text{ overlaps } i\} = \emptyset \).

- Then, the code dictates that \( \text{low}[j] \leq \text{m}[\text{left}][x] = \text{high}[j] \) for some \( j \in L \).
- Since \( j \in L \), it does not overlap \( i \), and hence \( \text{high}[i] < \text{low}[j] \).
- But, the binary-search-tree property implies that for all \( i' \in R \), we have \( \text{low}[j] \leq \text{low}[i'] \).
- But then \( \{i' \in R: i' \text{ overlaps } i\} = \emptyset \).

\[ \begin{array}{c}
\text{i} \\
\text{j} \\
\text{i'} \\
\end{array} \]

Combining info: Treap

- Heap and binary search tree properties together
- Treap
  - Red – heap
  - Blue – BST

Use rotations to “push up”

A geometric interpretation of the same treap.

Treap

- Tree + Heap = Treap
- BST + Heap of (key,priority) pair at the same time.
Treap

- Can be used to make a random-like tree: priorities can be assigned by random, unique values...
- In computer science, a treap is a binary search tree that orders the nodes by adding a random priority attribute to a node, as well as a key. The nodes are ordered so that the keys form a binary search tree and the priorities obey the max heap order property. The name treap is a portmanteau of tree and heap.
- A portmanteau word (pronounced /pɔːtˈmæn.təʊ/ (help·info)) is used broadly to mean a blend of two (or more) words and narrowly in linguistics fields to mean only a blend of two or more function words.

Bulk operations

- Union of two Treaps
- Intersection
- Set Difference
- These rely on two helper functions – split and merge

Split on k

- Insert (k, high_priority)
  - Left is smaller, right subtree larger than k

Union-find

- Domain X = \{ x_1, ..., x_n \}
- x_i belongs to a set S_j
- Non-intersecting sets.
- Union of sets: S_i' = S_i U S_j
- Find: Which set S_i does an element x_j belong to?

Sets = \{ \{1\}, \{2\}, ..., \{n\} \}
- Non-overlapping, each value belongs to a set
- Merge sets i, j (give new set id i, remove j)
  - Union
- Which set does x belong to ?
  - Find

![Diagram](image.png)

- Union = O(n)
- Find = O(1)

<table>
<thead>
<tr>
<th>Set Value</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set 1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>n</td>
</tr>
<tr>
<td>Set 2</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>n</td>
<td></td>
</tr>
<tr>
<td>Union 4, 6</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>n</td>
</tr>
<tr>
<td>Find = O(1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Union = O(n)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Link elements until “0”

<table>
<thead>
<tr>
<th>Set</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 0 0 3 0 0 0 6</td>
</tr>
</tbody>
</table>

Union 4, 6

$s_1 = \{3, 4\}$
$s_2 = \{6, 8\}$

<table>
<thead>
<tr>
<th>Set</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 0 0 3 0 4 0 6</td>
</tr>
</tbody>
</table>

Find = O(n)
Union = O(1)

- At every find – “flatten the tree”

Union-Find

A data structure for maintaining a collection of disjoint sets

Course: Data Structures
Lecturer: Uri Zwick
March 2008

Union-Find

- Make(x): Create a set containing x
- Union(x,y): Unite the sets containing x and y
- Find(x): Return a representative of the set containing x

Union Find

- make: O(1)
- union: O(\(\alpha(n)\))
- find: O(\(\alpha(n)\))
- Amortized
Fun applications: Generating mazes

<table>
<thead>
<tr>
<th>make(1)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>make(2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>make(16)</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>find(6)=find(7) ?</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>union(6,7)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>find(7)=find(11) ?</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>union(7,11)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Choose edges in random order and remove them if they connect two different regions

Generating mazes – a larger example

Construction time -- $O(n^2 \cdot \alpha(n^2))$

More serious applications:

- Maintaining an equivalence relation
- Incremental connectivity in graphs
- Computing minimum spanning trees
- …
Union Find
Represent each set as a rooted tree
Union by rank  Path compression

The parent of a vertex $x$ is denoted by $p[x]$
Find($x$) traces the path from $x$ to the root

Union by rank

Union by rank on its own gives $O(\log n)$ find time
A tree of rank $r$ contains at least $2^r$ elements
If $x$ is not a root, then $\text{rank}(x) = \text{rank}(p[x])$

Path Compression

Union Find - pseudocode

Function make-set($x$)
$p[x] \leftarrow x$
$\text{rank}[x] \leftarrow 0$

Function union($x$, $y$)
link(find($x$), find($y$))

Function link($x$, $y$)
if $\text{rank}[x] > \text{rank}[y]$ then
\[$p[x] \leftarrow y$
else
\[$p[y] \leftarrow x$
d if $\text{rank}[y] = \text{rank}[x]$ then
\[$\text{rank}[y] \leftarrow \text{rank}[y] + 1$

Function find($x$)
if $p[x] \neq x$ then
$\text{rank}[x] = \text{find}(p[x])$
return $p[x]$

Union-Find

Worst case

<table>
<thead>
<tr>
<th></th>
<th>make</th>
<th>link</th>
<th>find</th>
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</thead>
<tbody>
<tr>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(\log n)$</td>
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</table>

Amortized

<table>
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<tbody>
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<td>$O(1)$</td>
<td>$O(\alpha(n))$</td>
<td>$O(\alpha(n))$</td>
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</tbody>
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Nesting / Repeated application

$$f^{(i)}(n) = f(f(\ldots(f(n))\ldots))$$
$i$ times

$$f^{(0)}(n) = n$$

$$f^{(i)}(n) = f(f^{(i-1)}(n))$$, for $i > 0$

$$f(n) = n + 1$$
$$f^{(5)}(n) = n + 5$$

$$f(n) = 2n$$
$$f^{(2)}(n) = 2^{2n}$$

$$f(n) = \log n$$
$$f^{(2)}(n) = \log \log n$$
Ackermann’s function

\[ A_k(n) = \begin{cases} 
  n + 1 & \text{if } k = 1, \\
  A_{k-1}^{(n+1)}(n) & \text{if } k > 1.
\end{cases} \]

- \( A_1(n) = n + 1 \)
- \( A_2(n) = 2n + 1 \)
- \( A_3(n) = 2^{n+1}(n + 1) - 1 \)
- \( A_4(n) = ? \)

Ackermann’s function (modified)

\[ \tilde{A}_k(n) = \begin{cases} 
  n + 1 & \text{if } k = 1, \\
  \tilde{A}_{k-1}^{(n+1)}(n) & \text{if } k > 1.
\end{cases} \]

\[ \tilde{A}_2(n) = 2n \]
\[ \tilde{A}_3(n) = 2^n \]
\[ \tilde{A}_4(n) = \text{tower}(n) = 2^{2^{\ldots^{2}}} \uparrow n \]

Inverse functions

\[ F(n) \implies f(n) = \min\{k \geq 1 \mid F(k) \geq n\} \]

- \( F(n) = n + 1 \)
- \( f(n) = n - 1 \)
- \( F(n) = 2n \)
- \( f(n) = \lceil \frac{n}{2} \rceil \)
- \( F(n) = 2^n \)
- \( f(n) = \lceil \log_2 n \rceil \)
- \( F(n) = \text{tower}(n) \)
- \( f(n) = \log^* n \)

Inverse Ackermann function

\[ \alpha_r(n) = \min\{k \geq 1 \mid A_k(r) \geq n\} \]
\[ \alpha(n) = \alpha_1(n) = \min\{k \geq 1 \mid A_k(1) \geq n\} \]
\[ \alpha(n) \text{ is the inverse of the function } A_n(1) \]
\[ A_n(1) = A_{n-1}^{(2)}(1) = A_{n-1}(A_{n-1}(1)) > A_{n-1}(1) \]

Amortized cost of make

Actual cost: \( O(1) \)
\[ \Delta \Phi: \quad 0 \]

Amortized cost: \( O(1) \)

Amortized cost of link

Actual cost: \( O(1) \)

The potentials of \( y \) and \( z_1, \ldots, z_k \) can only decrease.

The potentials of \( x \) is increased by at most \( \alpha(n) \)

\[ \Delta \Phi \leq \alpha(n) \]

Actual cost: \( O(\alpha(n)) \)
Amortized cost of find

Amortized cost of find

Suppose that:

\[ 0 < i < j < \ell \]

\[ \text{level}(x_i) = \text{level}(x_j) \]

\[
A'_{\text{level}(x_i)}(\text{rank}[x_i]) \\
A'_{\text{level}(x_j)}(\text{rank}[x_i]) \\
A'_{\text{level}(x_j)}(\text{rank}[p[x_i]]) \\
A'_{\text{level}(x_j)}(\text{rank}[p[x_j]]) \\
\text{rank}[p[x_j]] \\
\text{rank}[x_j]
\]

\[ \text{is decreased} \]

Amortized cost of find

The only nodes that can retain their potential are:
- The first
- The last
- The last node of each level

Actual cost:

\[
\Delta \Phi \leq (\alpha(n) + 1) - (\ell + 1)
\]

Amortized cost:

\[ \alpha(n) + 1 \]