Algorithmics (6EAP)
MTAT.03.238
Linear structures, sorting, searching, etc
Jaak Vilo
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Big-Oh notation classes

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<td>( f(n) \in o(g(n)) )</td>
<td>f is dominated by g</td>
<td>Strictly below</td>
<td>&lt;</td>
</tr>
<tr>
<td>( f(n) \in O(g(n)) )</td>
<td>Bounded from above</td>
<td>Upper bound</td>
<td>( \leq )</td>
</tr>
<tr>
<td>( f(n) \in \Theta(g(n)) )</td>
<td>Bounded from above and below</td>
<td>“equal to”</td>
<td>=</td>
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<tr>
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Conclusions

• Algorithm complexity deals with the behavior in the long-term
  − worst case
  − average case
  − best case
  − typical
  − quite hard
  − bogus, “cheating”

• In practice, long-term sometimes not necessary
  − E.g. for sorting 20 elements, you don’t need fancy algorithms...

Linear, sequential, ordered, list ...

Memory, disk, tape etc – is an ordered sequentially addressed media.

Physical ordered list ~ array

• Memory /address/
  − Garbage collection

• Files (character/byte list/lines in text file,...)

• Disk
  − Disk fragmentation

Linear data structures: Arrays

• Array
• Bidirectional map
• Bit array
• Bit field
• Bitboard
• Bitmap
• Circular buffer
• Control table
• Image
• Dynamic array
• Gap buffer

• Hashed array tree
• Heightmap
• Lookup table
• Matrix
• Parallel array
• Sorted array
• Sparse array
• Sparse matrix
• Iliffe vector
• Variable-length array
Linear data structures: Lists

- Doubly linked list
- Array list
- Linked list
- Self-organizing list
- Skip list
- Unrolled linked list
- VList

Lists: Array

- Xor linked list
- Zipper
- Doubly connected edge list
- Difference list

Multiple lists, 2-D-arrays, etc...

2D array

Linear Lists

- Operations which one may want to perform on a linear list of \( n \) elements include:
  
  - gain access to the \( k \)th element of the list to examine and/or change the contents
  - insert a new element before or after the \( k \)th element
  - delete the \( k \)th element of the list

Abstract Data Type (ADT)

• High-level definition of data types
• An ADT specifies
  — A collection of data
  — A set of operations on the data or subsets of the data
• ADT does not specify how the operations should be implemented
• Examples:
  — vector, list, stack, queue, deque, priority queue, table (map), associative array, set, graph, digraph

ADT

• A datatype is a set of values and an associated set of operations
• A datatype is abstract if it is completely described by its set of operations regardless of its implementation
• This means that it is possible to change the implementation of the datatype without changing its use
• The datatype is thus described by a set of procedures
• These operations are the only thing that a user of the abstraction can assume

Primitive & composite types

Primitive types
• Boolean (for boolean values True/False)
• Char (for character values)
• int (for integral or fixed-precision values)
• float (for storing real number values)
• Double (a larger size of type float)
• String (for string of chars)
• Enumerated type

Composite types
• Array
• Record (also called tuple or struct)
• Union
• Tagged union (also called a variant, variant record, discriminated union, or disjoint union)
• Plain old data structure

Abstract Data Types (ADT)

• Some common ADTs, which have proved useful in a great variety of applications, are

<table>
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<th>Stack</th>
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<tbody>
<tr>
<td>List</td>
<td>Graph</td>
</tr>
<tr>
<td>Set</td>
<td>Queue</td>
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<tr>
<td>Multiset</td>
<td>Priority queue</td>
</tr>
<tr>
<td>Map</td>
<td>Double-ended queue</td>
</tr>
<tr>
<td>Multimap</td>
<td>Double-ended priority queue</td>
</tr>
</tbody>
</table>

• Each of these ADTs may be defined in many ways and variants, not necessarily equivalent.

Abstract data types:

• Dictionary (key,value)
• Stack (LIFO)
• Queue (FIFO)
• Queue (double-ended)
• Priority queue (fetch highest-priority object)
• ...

Dictionary

• Container of key-element (k,e) pairs
• Required operations:
  — insert( k,e ),
  — remove( k ),
  — find( k ),
  — isEmpty()
• May also support (when an order is provided):
  — closestKeyBefore( k ),
  — closestElemAfter( k )
• Note: No duplicate keys
Some data structures for Dictionary ADT

- **Unordered**
  - Array
  - Sequence/list

- **Ordered**
  - Array
  - Sequence (Skip Lists)
  - Binary Search Tree (BST)
  - AVL trees, red-black trees
  - (2; 4) Trees
  - B-Trees

- **Valued**
  - Hash Tables
  - Extendible Hashing

**Linear data structures**

- **Arrays**
  - Hashed array tree
  - Heightmap
  - Lookup table
  - Matrix
  - Parallel array
  - Sorted array
  - Sparse array
  - Sparse matrix
  - Sparse vector
  - Variable-length array

- **Lists**
  - Doubly linked list
  - Linked list
  - Self-organizing list
  - Skip list
  - Unrolled linked list
  - VList
  - Xor linked list
  - Zipper
  - Doubly connected edge list

**Trees ...**

- Binary tree
- AVL tree
- Balanced tree
- Red Black tree
- Splay tree
- Heap tree
- Dual tree
- Threaded binary tree
- Top tree

- Self heap
- Height heap
- Light heap
- Triple
- Triple heap
- Triple tree
- Red-Black tree
- Splay tree
- Binary heap
- Red-Black heap
- Multiheap
- Stack tree
- Threaded binary tree
- Top tree

**Hashes, Graphs, Other**

- **Hashes**
  - Bloom filter
  - Distributed hash table
  - Hash array mapped trie
  - Hash list
  - Hash table
  - Hash tree
  - Hash trie
  - Koonde
  - Prefix hash tree

- **Graphs**
  - Adjacency list
  - Adjacency matrix
  - Graph-structured stack
  - Scene graph
  - Binary decision diagram
  - Zero suppressed decision diagram
  - And-inverter graph
  - Directed graph
  - Directed acyclic graph

- **Other**
  - Propositional directed acyclic graph
  - Multigraph
  - Hypergraph

**Lists: Array**

- **Access i** \(O(1)\)
- **Insert to end** \(O(1)\)
- **Delete from end** \(O(n)\)
- **Insert** \(O(n)\)
- **Delete** \(O(n)\)
- **Search** \(O(n)\)

- **Lists: Array**
  - **Access i** \(O(1)\)
  - **Insert to end** \(O(1)\)
  - **Delete from end** \(O(n)\)
  - **Insert** \(O(n)\)
  - **Delete** \(O(n)\)
  - **Search** \(O(n)\)
Linear Lists

- Other operations on a linear list may include:
  - determine the number of elements
  - search the list
  - sort a list
  - combine two or more linear lists
  - split a linear list into two or more lists
  - make a copy of a list

Stack

- push(x) -- add to end (add to top)
- pop() -- fetch from end (top)

- O(1) in all reasonable cases 😊
- LIFO – Last In, First Out

Linked lists

Linked lists: add

Linked lists: delete (+ garbage collection?)

Operations

- Array indexed from 0 to n – 1:

<table>
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<th>(k=1)</th>
<th>(1 &lt; k &lt; n)</th>
<th>(k=n)</th>
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- Singly-linked list with head and tail pointers

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  under the assumption we have a pointer to the kth node, \(O(1)\) otherwise
Improving Run-Time Efficiency

- We can improve the run-time efficiency of a linked list by using a doubly-linked list:

  **Singly-linked list:**
  ```
  list_head → 1 → 2 → 3 → ... → n → list_tail
  ```

  **Doubly-linked list:**
  ```
  list_head → 1 ← 2 ← 3 ← ... ← n ← list_tail
  ```

  - Improvements at operations requiring access to the previous node
  - Increases memory requirements...

---

Improving Efficiency

**Singly-linked list:**

- Access/change the kth element: $O(n)$
- Insert before or after the kth element: $O(n)$
- Delete the kth element: $O(n)$

**Doubly-linked list:**

- Access/change the kth element: $O(1)$
- Insert before or after the kth element: $O(1)$
- Delete the kth element: $O(1)$

---

Introduction to linked lists

- Consider the following struct definition
  ```
  struct node {
      string word;
      int num;
      node *next;
  };
  ```

- **Node creation:**
  ```
  node *p = new node;
  ```

- **Node initialization:**
  ```
  p->num = 5;
  p->word = "Ali";
  ```

- **Node connection:**
  ```
  p->next = NULL;
  ```

---

Introduction to linked lists: inserting a node

- Node creation:
  ```
  node *p;
  p = new node;
  ```

- Node initialization:
  ```
  p->num = 5;
  p->word = "Ali";
  ```

- Node connection:
  ```
  p->next = NULL;
  ```

---

Introduction to linked lists: adding a new node

- How can you add another node that is pointed by p->link?
  ```
  node *q;
  ```

- Node creation:
  ```
  node *p;
  ```

- Node initialization:
  ```
  p = new node;
  ```

- Node connection:
  ```
  p->num = 5;
  p->word = "Ali";
  ```

- Node connection:
  ```
  p->next = NULL;
  ```

- Node connection:
  ```
  node *q;
  ```

---
**Introduction to linked lists**

```c
node *p;
p = new node;
p->num = 5;
p->word = "Ali";
p->next = NULL;

node *q;
q = new node;
```

- Allocate memory for a new node.
- Assign values to the node.
- Link to the next node.

**Pointers in C/C++**

```c
p = new node; delete p;
p = new node[20];
p = malloc( sizeof( node ) ); free p;
p = malloc( sizeof( node ) * 20 ); (p+10)->next = NULL; /* 11th elements */
```

- Use `malloc` and `free` for dynamic memory allocation and deallocation.
- Allocate multiple nodes at once.
- Link new nodes.

**Book-keeping**

- `malloc`, `new` — "remember" what has been created `free(p), delete` (C/C++)
- When you need many small areas to be allocated, reserve a big chunk (array) and maintain your own set of free objects
- Elements of array of objects can be pointed by the pointer to an object.

**Object**

- `Object = new object_type ;`
- Equals to creating a new object with necessary size of allocated memory (delete can free it)
Some links

- C++ Memory Management: What is the difference between malloc/free and new/delete?

Alternative: **arrays and integers**

- If you want to test pointers and linked list etc. data structures, but do not have pointers familiar (yet)
- Use arrays and indexes to array elements instead...

Replacing pointers with array index

Maintenance list of free objects

Multiple lists, single free list

Hack: allocate more arrays ...
Queue

- enqueue(x) - add to end
- dequeue() - fetch from beginning
- FIFO – First In First Out
- O(1) in all reasonable cases

Circular buffer

- A circular buffer or ring buffer is a data structure that uses a single, fixed-size buffer as if it were connected end-to-end. This structure lends itself easily to buffering data streams.
Stack

• push(x) -- add to end (add to top)
• pop() -- fetch from end (top)

• O(1) in all reasonable cases 😊
• LIFO – Last In, First Out

Stack based languages

• Implement a postfix calculator
  – Reverse Polish notation
  
  5 4 3 * 2 + => 5+(4*3)-2

• Very simple to parse and interpret
• FORTH, Postscript are stack-based languages

Array based stack

• How to know how big a stack shall ever be?

3 6 7 5
3 6 7 5 2

• When full, allocate bigger table dynamically, and copy all previous values there

• O(n) ?

Array based stack

• When full, create 2x bigger table, copy previous n elements:

• After every 2^k insertions, perform O(n) copy

• O(n) individual insertions +
• n/2 + n/4 + n/8 ... copy-ing
• Total: O(n) effort!

What about deletions?

• when n=32 -> 33 (copy 32, insert 1)
• delete: 33->32
  -- should you delete (slots 33-64) immediately?

What about deletions?

• when n=32 -> 33 (copy 32, insert 1)
• delete: 33->32
  -- should you delete immediately?
  -- Delete only when becomes less than 1/4th full
  -- Have to delete at least n/2 to decrease
  -- Have to add at least n to increase size
  -- Most operations, O(1) effort
  -- But few operations take O(n) to copy
  -- For any m operations, O(m) time
Amortized analysis

- Analyze the time complexity over the entire “lifespan” of the algorithm
- Some operations that cost more will be “covered” by many other operations taking less

Lists and dictionary ADT...

- How to maintain a dictionary using (linked) lists?
  - Is k in D?
    - go through all elements d of D, test if d==k \( O(n) \)
    - If sorted: d= first(D); while( d<k ) d=next(D);
    - on average \( n/2 \) tests ...
- \( \text{Add}(k,D) \Rightarrow \text{insert}(k,D) = O(1) \) or \( O(n) \) – test for uniqueness

Array based sorted list

- is d in D?
- Binary search in D

Binary search – recursive

```c
BinarySearch(A[0..N-1], value, low, high)
{
    if (high < low)
        return -1 // not found
    mid = (low + high) / 2 // Note: not (low + high) / 2 !!
    if (A[mid] > value)
        return BinarySearch(A, value, low, mid-1)
    else if (A[mid] < value)
        return BinarySearch(A, value, mid+1, high)
    else
        return mid // found
}
```

Binary search – iterative

```c
BinarySearch(A[0..N-1], value)
{
    low = 0; high = N - 1;
    while (low <= high)
    {
        mid = (low + high) / 2 // Note: not (low + high) / 2 !!
        if (A[mid] > value)
            high = mid - 1
        else if (A[mid] < value)
            low = mid + 1
        else
            return mid // found
    }
    return -1 // not found
}
```
Work performed

- \( x \leftrightarrow A[18] \) ? <
- \( x \leftrightarrow A[9] \) ? >
- \( x \leftrightarrow A[13] \) ? ==

- \( O(\lg n) \)

Sorting

- given a list, arrange values so that \( L[1] <= L[2] <= ... <= L[n] \)
- \( n \) elements \( \Rightarrow n! \) possible orderings
- One test \( L[i] <= L[j] \) can divide \( n! \) to 2
  - Make a binary tree and calculate the depth
- \( \log(n!) = \Omega(n \log n) \)
- Hence, lower bound for sorting is \( \Omega(n \log n) \)

Decision-tree example

Sort \( (a_1, a_2, a_3) \) = \( (9, 4, 6) \):

Each leaf contains a permutation \( (\pi(1), \pi(2), ... , \pi(n)) \) to indicate that the ordering \( a_{\pi(1)} \leq a_{\pi(2)} \leq ... \leq a_{\pi(n)} \) has been established.

Decision tree model

- \( n! \) orderings (leaves)
- Height of such tree?

Proof: \( \log(n!) = \Omega(n \log n) \)

- \( \log(n!) = \log n + \log(n-1) + \log(n-2) + ... + \log(1) \)
  \[ \geq \frac{n}{2} \log \left( \frac{n}{2} \right) \]
  \[ = \Omega(n \log n) \]

Lower bound for decision-tree sorting

**Theorem.** Any decision tree that can sort \( n \) elements must have height \( \Omega(n \log n) \).

**Proof.** The tree must contain \( \geq n! \) leaves, since there are \( n! \) possible permutations. A height-\( h \) binary tree has \( \leq 2^h \) leaves. Thus, \( n! \leq 2^h \).

\[ \therefore h \geq \log(n!) \quad \text{(lg is mono. increasing)} \]
\[ \geq \log \left( \frac{n!}{e^n} \right) \quad \text{(Stirling’s formula)} \]
\[ = n \log n - n \log e \]
\[ = \Omega(n \log n) \]
The divide-and-conquer design paradigm

1. Divide the problem (instance) into subproblems.
2. Conquer the subproblems by solving them recursively.
3. Combine subproblem solutions.

\[
\left\lceil \log_2 n! \right\rceil \geq \log_2 n! \\
\geq \sum_{i=1}^{n} \log_2 i \\
\geq \sum_{i=1}^{n/2} \log_2 n/2 \\
\geq \frac{n}{2} \log_2 n/2 \\
= \Omega(n \log n).
\]

Merge sort

\[
\text{Merge-Sort}(A, p, r) \\
\text{if } p < r \\
\quad \text{then } q = \lfloor (p+r)/2 \rfloor \\
\quad \text{// floor} \\
\quad \text{Merge-Sort}(A, p, q) \\
\quad \text{Merge-Sort}(A, q+1, r) \\
\quad \text{Merge}(A, p, q, r)
\]

It was invented by John von Neumann in 1945.

Example

• Applying the merge sort algorithm:

  Merge of two lists: \( \Theta(n) \)

A, B – lists to be merged
L = new list; // empty
while( A not empty \ and \ B not empty )
  if A.first() <= B.first() 
    then append( L, A.first() ) ; A = rest(A) ;
   else append( L, B.first() ) ; B = rest(B) ;
append( L, A); // all remaining elements of A
append( L, B ); // all remaining elements of B
return L

Wikipedia / viz.
Run-time Analysis of Merge Sort

- Thus, the time required to sort an array of size $n > 1$ is:
  - the time required to sort the first half,
  - the time required to sort the second half, and
  - the time required to merge the two lists
- That is:
  \[
  T(n) = \begin{cases} 
  \Theta(1) & n = 1 \\
  2T\left(\frac{n}{2}\right) + \Theta(n) & n > 1 
  \end{cases}
  \]

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

\[
\begin{array}{c}
  T(n) = \Theta(n) \\
  \Theta(1) \quad \text{#leaves} = n \\
  \Theta(n) \quad \text{Total} = \Theta(n \log n)
\end{array}
\]

Merge sort

- Worst case, average case, best case ... $\Theta(n \log n)$
- Common wisdom:
  - Requires additional space for merging (in case of arrays)
- Homework*: develop in-place merge of two lists implemented in arrays /compare speed/

Quicksort

- Divide-and-conquer algorithm.
- Sorts “in-place” (like insertion sort, but not like merge sort).
- Very practical (with tuning).

Divide and conquer

Quicksort an $n$-element array:

1. Divide: Partition the array into two subarrays around a pivot $x$ such that elements in lower subarray $\leq x$ and elements in upper subarray $\geq x$.

2. Conquer: Recursively sort the two subarrays.


Key: Linear-time partitioning subroutine.
Pseudocode for quicksort

\[\text{QUICKSORT}(A, p, r)\]
\[\text{if } p < r \quad \text{then } q \leftarrow \text{PARTITION}(A, p, r)\]
\[\text{QUICKSORT}(A, p, q - 1)\]
\[\text{QUICKSORT}(A, q + 1, r)\]

Initial call: \(\text{QUICKSORT}(A, 1, n)\)

Partitioning subroutine

\[\text{PARTITION}(A, p, q) \uparrow A[p \ldots q]\]
\[x \leftarrow A[p]\]
\[\text{pivot} = A[p]\]
\[i \leftarrow p\]
\[\text{for } j \leftarrow p + 1 \text{ to } q\]
\[\text{do if } A[j] \leq x \quad \text{then } i \leftarrow i + 1\]
\[\text{exchange } A[i] \leftrightarrow A[j]\]
\[\text{return } i\]

Invariant: \(x \leq i \leq j \leq q\)

Running time \(= O(n)\) for \(n\) elements.

Wikipedia / “video”

Partitioning version 2

\[\text{pivot} = A[R];\]
\[i = L; j = R - 1;\]
\[\text{while } (i < j) \quad \text{while } (A[i] < \text{pivot}) \quad i++; // \text{ will stop at pivot latest}\]
\[\text{while } (i < j \text{ and } A[j] >= \text{ pivot}) \quad j--;\]
\[\text{if } (i < j) \quad \{ \text{ swap } (A[i], A[j]); \quad i++; \quad j--; \}\]
\[A[R] = A[i];\]
\[A[i] = \text{pivot};\]
\[\text{return } i;\]

Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

\[T(n) = T(0) + T(n - 1) + \Theta(n)\]
\[= \Theta(1) + T(n - 1) + \Theta(n)\]
\[= T(n - 1) + \Theta(n)\]
\[= \Theta(n^2) \quad \text{(arithmetic series)}\]

Best-case analysis

(For intuition only!)

If we’re lucky, \(\text{PARTITION}\) splits the array evenly:

\[T(n) = 2T(n/2) + \Theta(n)\]
\[= \Theta(n \log n) \quad \text{(same as merge sort)}\]

What if the split is always \(\frac{1}{10}, \frac{9}{10}\)?

\[T(n) = T(\lfloor \frac{n}{10} \rfloor) + T(\lceil \frac{n}{10} \rceil) + \Theta(n)\]

What is the solution to this recurrence?
Choice of pivot in Quicksort

- Select median of three ...
- Select random – opponent can not choose the winning strategy against you!

More intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ...

\[ L(n) = 2U(n/2) + \Theta(n) \quad \text{lucky} \]

\[ U(n) = L(n-1) + \Theta(n) \quad \text{unlucky} \]

Solving:

\[ L(n) = 2(L(n/2 - 1) + \Theta(n/2)) + \Theta(n) \]
\[ = 2L(n/2 - 1) + \Theta(n) \]
\[ = \Theta(n \log n) \quad \text{Lucky!} \]

How can we make sure we are usually lucky?

Randomized quicksort

**IDEA**: Partition around a random element.
- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

Random pivot

Select pivot randomly from the region (blue) and swap with last position
Select pivot as a median of 3 [or more] random values from region
Apply non-recursive sort for array less than 10-20
• 2-pivot version of Quicksort
  – (split in 3 regions!)
• Handle equal values (equal to pivots)

**Bentley-McIlroy 3-way partitioning**

<table>
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<tr>
<td>equal</td>
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</table>

- move from left to find an element that is not less
- move from right to find an element that is not greater
- stop if pointers have crossed
- exchange if left element equal, exchange to left end
  - if right element equal, exchange to right end
- Swap equals to center after partition

**KEY FEATURES**
- always uses N-1 (three-way) compares
- no extra overhead if no equal keys
- only one "extra" exchange per equal key

Robert Sedgewick
Jan Bentley

**Quicksort with 3-way partitioning**

```c
void quicksort(int a[], int l, int r)
int i = l-1, j = r, p = l-1, q = r; Item v = a[r];
if (r <= l return;
for (; ; )
{
while (v < a[p+1]) break;
if (i >= j break;
each(a[i], a[j]);
if (a[i] == v) p++;
each(a[p], a[i]);
}if (v == a[j]) q--;
each(a[q], a[j]);
}each(a[i], a[r]); j = i-1; i = i+1;
for (k = l; k < p; k++) each(a[k], a[i]);
for (k = r-1; k > q; k--) each(a[k], a[q]);
quicksort(a, i, j);
quicksort(a, i, r);
```

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**Quicksort in practice**

• Quicksort is a great general-purpose sorting algorithm.
• Quicksort is typically over twice as fast as merge sort.
• Quicksort can benefit substantially from code tuning.
• Quicksort behaves well even with caching and virtual memory.

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**Randomized quicksort analysis**

\[
T(n) = \begin{cases}
T(0) + T(n-1) + \Theta(n) & \text{if } 0:n-1 \text{ split}, \\
T(1) + T(n-2) + \Theta(n) & \text{if } 1:n-2 \text{ split}, \\
\vdots \\
T(n-1) + T(0) + \Theta(n) & \text{if } n-1:0 \text{ split},
\end{cases}
\]

**Randomized quicksort analysis**

Let \( T(n) \) = the random variable for the running time of randomized quicksort on an input of size \( n \), assuming random numbers are independent.

For \( k = 0, 1, \ldots, n-1 \), define the *indicator random variable*

\[
X_k = \begin{cases}
1 & \text{if } \text{PARTITION generates a } k:n-k-1 \text{ split}, \\
0 & \text{otherwise}.
\end{cases}
\]

\[
\mathbb{E}[X_k] = \mathbb{P}(X_k = 1) = \frac{1}{n}, \text{ since all splits are equally likely, assuming elements are distinct.}
\]

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### Analysis (continued)

\[
T(n) = \begin{cases} 
T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split}, \\
T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split}, \\
T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split}, \\
\end{cases}
\]

\[
= \sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n))
\]

### Calculating expectation

\[
E[T(n)] = E\left[ \sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)) \right]
\]

Linearity of expectation.

\[
= \sum_{k=0}^{n-1} E[X_k (T(k) + T(n-k-1) + \Theta(n))]
\]

Independence of \(X_k\) from other random choices.

### Calculating expectation

\[
E[T(n)] = E\left[ \sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)) \right]
\]

\[
= \sum_{k=0}^{n-1} E[X_k (T(k) + T(n-k-1) + \Theta(n))]
\]

\[
= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]
\]

\[
= \sum_{k=0}^{n-1} E[X_k] \cdot \left( E[T(k)] + E[T(n-k-1)] + \Theta(n) \right)
\]

Linearity of expectation; \(E[X_k] = 1/n\).

### Calculating expectation

\[
E[T(n)] = E\left[ \sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)) \right]
\]

\[
= \sum_{k=0}^{n-1} E[X_k (T(k) + T(n-k-1) + \Theta(n))]
\]

\[
= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]
\]

\[
= \sum_{k=0}^{n-1} E[X_k] \cdot \left( E[T(k)] + E[T(n-k-1)] + \Theta(n) \right)
\]

\[
= \sum_{k=0}^{n-1} \frac{1}{n} E[T(k)] + \sum_{k=0}^{n-1} \frac{1}{n} E[T(n-k-1)] + \sum_{n=0}^{\infty} \Theta(n)
\]

Summations have identical terms.
Hairy recurrence

\[ E[T(n)] = \sum_{k=2}^{n-1} E[T(k)] + \Theta(n) \]

(The \( k = 0, 1 \) terms can be absorbed in the \( \Theta(n) \).)

**Prove:** \( E[T(n)] \leq a n \log n \) for constant \( a > 0 \).

- Choose \( a \) large enough so that \( a n \log n \)
  dominates \( E[T(n)] \) for sufficiently small \( n \geq 2 \).

**Use fact:** \( \sum_{k=2}^{n-1} k \log k \leq \frac{1}{2} n^2 \log n - \frac{1}{8} n^2 \) (exercise).

Substitution method

\[ E[T(n)] \leq 2 \sum_{k=2}^{n-1} ak \log k + \Theta(n) \]

\[ = 2a \left( \frac{1}{n} n^2 \log n - \frac{1}{8} n^2 \right) + \Theta(n) \]

\[ = an \log n \left( \frac{a n}{4} - \Theta(n) \right) \]

\[ \leq an \log n , \]

if \( a \) is chosen large enough so that \( an/4 \) dominates the \( \Theta(n) \).

Alternative materials

- QuickSort average case analysis
  - [http://eid.ee/10z](http://eid.ee/10z)
  - [https://secweb.cs.odu.edu/~zeil/cs361/web/website/Lectures/quick/pages/ar01s05.html](https://secweb.cs.odu.edu/~zeil/cs361/web/website/Lectures/quick/pages/ar01s05.html)
- [http://eid.ee/10y](http://eid.ee/10y) - MIT Open Courseware -
  Asymptotic notation, Recurrences, Substitution Master Method

The master method

The master method applies to recurrences of the form

\[ T(n) = a T(n/b) + f(n) , \]

where \( a \geq 1, b > 1 \), and \( f(n) \) is asymptotically positive.

\[ T(n) = a T(n/b) + f(n) \]

\[ n^{\log_b a} \text{ vs } f(n) \]
Three common cases

Compare \( f(n) \) with \( n^\log_a b \):

1. \( f(n) = O(n^\log_a b - \varepsilon) \) for some constant \( \varepsilon > 0 \).
   - \( f(n) \) grows polynomially slower than \( n^\log_a b \) (by an \( n^\varepsilon \) factor).
   \textbf{Solution:} \( T(n) = \Theta(n^\log_a b) \).

Three common cases (cont.)

Compare \( f(n) \) with \( n^{\log_a b} \):

3. \( f(n) = \Omega(n^{\log_a b + \varepsilon}) \) for some constant \( \varepsilon > 0 \).
   - \( f(n) \) grows polynomially faster than \( n^\log_a b \) (by an \( n^\varepsilon \) factor),
   \textbf{and} \( f(n) \) satisfies the \textit{regularity condition} that \( af(n/b) \leq cf(n) \) for some constant \( c < 1 \).
   \textbf{Solution:} \( T(n) = \Theta(f(n)) \).

Examples

\textbf{Ex.} \( T(n) = 4T(n/2) + n^3 \)

\( a = 4, b = 2 \Rightarrow n^{\log_a b} = n^2 \); \( f(n) = n^3 \).

\textbf{Case 3:} \( f(n) = \Omega(n^{2 - \varepsilon}) \) for \( \varepsilon = 1 \)
\textit{and} \( 4(n/2)^3 \leq cn^3 \) (reg. cond.) for \( c = 1/2 \).
\therefore \( T(n) = \Theta(n^3) \).
**Idea of master theorem**

**Recursion tree:**

```
  f(n)  
  /   
/     \  
/       
  a    f(n/b)  
  |      /   \   
  |     /     \   
  |    /       \   
  |   /         \   
  |  /           \   
  | /            \   
  |/              \   
  f(n/b^2)    f(n/b^2)   f(n/b^2)   
```

- **Case 1:** The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.
- **Case 2:** The weight is approximately the same on each of the \( \log n \) levels.
- **Case 3:** The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.
Back to sorting

We can sort in $O(n \log n)$

- Is that the best we can do?
- Remember: using comparisons $<$, $>$, $\leq$, $\geq$ we cannot do better than $O(n \log n)$

How fast can we sort $n$ integers?

- Sort people by their sex? (F/M, 0/1)
- Sort people by year of birth?

Sorting in linear time

**Counting sort**: No comparisons between elements.

- Input: $A[1 \ldots n]$, where $A[j] \in \{1, 2, \ldots, k\}$.
- Output: $B[1 \ldots n]$, sorted.
- Auxiliary storage: $C[1 \ldots k]$.

**Counting sort**

```plaintext
for i ← 1 to k
  do C[i] ← 0
for j ← 1 to n
  do C[A[j]] ← C[A[j]] + 1  ▷ C[i] = |{key = i}|
for i ← 2 to k
  do C[i] ← C[i] + C[i−1]   ▷ C[i] = |{key ≤ i}|  
for j ← n downto 1
  do B[C[A[j]]] ← A[j]
     C[A[j]] ← C[A[j]] − 1
```

**Loop 1**

- $A$: 1 2 3 4 5
- $B$:  
- $C$: 0 0 0 0

- for $i ← 1$ to $k$
  do $C[i] ← 0$
Loop 2

\[ A: \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 4 & 3 \end{array} \quad C: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 2 \end{array} \]

for \( j \leftarrow 1 \) to \( n \)
\[ \text{do } C[A[j]]=C[A[j]]+1 \quad \text{and } C'[i]=|\{ \text{key} = i \}| \]

Loop 3

\[ A: \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 4 & 3 \end{array} \quad C: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 2 \end{array} \]

for \( i \leftarrow 2 \) to \( k \)
\[ \text{do } C'[i]=C'[i]+C'[i-1] \quad \text{and } C'[i]=|\{ \text{key} \leq i \}| \]

Loop 4

\[ A: \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 4 & 3 \end{array} \quad C: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 5 \end{array} \]

for \( j \leftarrow n \) downto \( 1 \)
\[ \text{do } B[C[A[j]]]=A[j] \]
\[ C[A[j]]\leftarrow C[A[j]]-1 \]

C': \[ \begin{array}{cccc} 1 & 1 & 2 & 4 \end{array} \]

Analysis

\[ \Theta(k) \quad \text{for } i \leftarrow 1 \) to \( k \)
\[ \text{do } C'[i]=0 \]
\[ \Theta(n) \quad \text{for } j \leftarrow 1 \) to \( n \)
\[ \text{do } C[A[j]]=C[A[j]]+1 \]
\[ \Theta(k) \quad \text{for } i \leftarrow 2 \) to \( k \)
\[ \text{do } C'[i]=C'[i]+C'[i-1] \]
\[ \Theta(n) \quad \text{for } j \leftarrow n \) downto \( 1 \)
\[ \text{do } B[C[A[j]]]=A[j] \]
\[ C[A[j]]\leftarrow C[A[j]]-1 \]
\[ \Theta(n+k) \]

Running time

If \( k = \Omega(n) \), then counting sort takes \( \Theta(n) \) time.
- But, sorting takes \( \Omega(n \ lg \ n) \) time!
- Where’s the fallacy?

Answer:
- Comparison sorting takes \( \Omega(n \ lg \ n) \) time.
- Counting sort is not a comparison sort.
- In fact, not a single comparison between elements occurs!

Stable sorting

Counting sort is a **stable** sort: it preserves the input order among equal elements.

\[ A: \begin{array}{cccc} 4 & 1 & 3 & 4 & 3 \end{array} \]

\[ B: \begin{array}{cccc} 1 & 3 & 4 & 3 \end{array} \]

Exercise: What other sorts have this property?
Radix sort

- **Origin**: Herman Hollerith’s card-sorting machine for the 1890 U.S. Census. (See Appendix B.)
- Digit-by-digit sort.
- Hollerith’s original (bad) idea: sort on most-significant digit first.
- Good idea: Sort on *least-significant digit first* with auxiliary *stable* sort.

Radix sort

Radix-Sort(A,d)
1. for $i = 1$ to $d$ /* least significant to most significant */
2. use a stable sort to sort $A$ on digit $i$

Operation of radix sort

<table>
<thead>
<tr>
<th>329</th>
<th>720</th>
<th>720</th>
<th>329</th>
</tr>
</thead>
<tbody>
<tr>
<td>457</td>
<td>355</td>
<td>329</td>
<td>355</td>
</tr>
<tr>
<td>657</td>
<td>436</td>
<td>436</td>
<td>436</td>
</tr>
<tr>
<td>839</td>
<td>457</td>
<td>839</td>
<td>457</td>
</tr>
<tr>
<td>436</td>
<td>657</td>
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<td>720</td>
<td>329</td>
<td>457</td>
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</tr>
<tr>
<td>355</td>
<td>839</td>
<td>657</td>
<td>839</td>
</tr>
</tbody>
</table>

Correctness of radix sort

**Induction on digit position**

- Assume that the numbers are sorted by their low-order $t-1$ digits.
- Sort on digit $t$
  - Two numbers that differ in digit $t$ are correctly sorted.
  - Two numbers equal in digit $t$ are put in the same order as the input ⇒ correct order.

Analysis of radix sort

- Assume counting sort is the auxiliary stable sort.
- Sort $n$ computer words of $b$ bits each.
- Each word can be viewed as having $b/r$ base-$2^r$ digits.

**Example**: 32-bit word

```
8 8 8 8
```

$r = 8 \Rightarrow b/r = 4$ passes of counting sort on base-$2^8$ digits; or $r = 16 \Rightarrow b/r = 2$ passes of counting sort on base-$2^{16}$ digits.

*How many passes should we make?*
Analysis (continued)

Recall: Counting sort takes $\Theta(n + k)$ time to sort $n$ numbers in the range from 0 to $k - 1$. If each $b$-bit word is broken into $r$-bit pieces, each pass of counting sort takes $\Theta(n + 2^r)$ time. Since there are $b/r$ passes, we have

$$T(n, b) = \Theta\left(\frac{b}{r} \left(n + 2^r\right)\right).$$

Choose $r$ to minimize $T(n, b)$:
- Increasing $r$ means fewer passes, but as $r \gg \lg n$, the time grows exponentially.

Choosing $r$:

$$T(n, b) = \Theta\left(\frac{b}{r} \left(n + 2^r\right)\right)$$

Minimize $T(n, b)$ by differentiating and setting to 0. Or, just observe that we don’t want $2^r \gg n$, and there’s no harm asymptotically in choosing $r$ as large as possible subject to this constraint.

Choosing $r = \lg n$ implies $T(n, b) = \Theta(bn/\lg n)$.
- For numbers in the range from 0 to $n^d - 1$, we have $b = d\lg n \implies$ radix sort runs in $\Theta(dn)$ time.

Conclusions

In practice, radix sort is fast for large inputs, as well as simple to code and maintain.

Example (32-bit numbers):
- At most 3 passes when sorting $\geq 2000$ numbers.
- Merge sort and quicksort do at least $\lceil \lg 2000 \rceil = 11$ passes.

Downside: Unlike quicksort, radix sort displays little locality of reference, and thus a well-tuned quicksort fares better on modern processors, which feature steep memory hierarchies.

Radix sort using lists (stable)
Why not from left to right?

- Swap ‘0’ with first ‘1’
- Idea 1: recursively sort first and second half
  - Exercise?

Bitwise sort left to right

- Idea 2:
  - swap elements only if the prefixes match...
  - For all bits from most significant
    - advance when 0
    - when 1 look for next 0
      - if prefix matches, swap
      - otherwise keep advancing on 0’s and look for next 1

Bitwise sort left to right

- Idea 2:
  - swap elements only if the prefixes match...
  - For all bits from most significant
    - advance when 0
    - when 1 look for next 0
      - if prefix matches, swap
      - otherwise keep advancing on 0’s and look for next 1

Bucket sort

- Assume uniform distribution
- Allocate O(n) buckets
- Assign each value to pre-assigned bucket

Bitwise from left to right

- Swap ‘0’ with first ‘1’
Sort small buckets with insertion sort

http://sortbenchmark.org/

• The sort input records must be 100 bytes in length, with the first 10 bytes being a random key
• Minutesort – max amount sorted in 1 minute
  – 116GB in 58.7 sec (Jim Wyllie, IBM Research)
  – 40-node 80-Itanium cluster, SAN array of 2,520 disks
• 2009, 500 GB Hadoop 1406 nodes x (2 Quadcore Xeons, 8 GB memory, 4 SATA)
  Owen O’Malley and Arun Murthy
  Yahoo Inc.
• Performance / Price Sort and PennySort
Sort Benchmark

- http://sortbenchmark.org/
- Sort Benchmark Home Page
  - We have a new benchmark called GraySort, in memory of the father of the sort benchmarks, Jim Gray. It replaces TerabyteSort which is now retired.
  - Unlike 2010, we will not be accepting early entries for the 2011 year. The deadline for submitting entries is April 1, 2011.
  - All hardware used must be off-the-shelf and unmodified.
  - For Daytona cluster sorts where input sampling is used to determine the output partition boundaries, the input sampling must be done evenly across all input partitions.
  
  New rules for GraySort:
  - The input file size is now minimum ~100TB or 1T records. Entries with larger input sizes also qualify.
  - The winner will have the fastest SortedRecs/Min.
  - We now provide a new input generator that works in parallel and generates binary data. See below.
  - For the Daytona category, we have two new requirements. (1) The sort must run continuously (uninterupted) for a minimum 1 hour. (This is a minimum reliability requirement). (2) The system cannot overwrite the input file.

Order statistics

- Minimum – the smallest value
- Maximum – the largest value
- In general i’th value.
- Find the median of the values in the array
- Median in sorted array A:
  - n is odd \( A\left(\frac{n+1}{2}\right) \)
  - n is even \( A\left(\frac{n+1}{2}\right) \) or \( A\left(\frac{n+1}{2}\right) \)

Q: Find i’th value in unsorted data

A. \( O(n) \)
B. \( O(n \log \log n) \)
C. \( O(n \log n) \)
D. \( O(n \log^2 n) \)

Minimum

\[
\text{Minimum}(A) \\
1 \quad \text{min} = A[1] \\
2 \quad \text{for } i = 2 \text{ to length}(A) \\
3 \quad \text{if } \text{min} > A[i] \\
4 \quad \text{then } \text{min} = A[i] \\
5 \quad \text{return min}
\]

n-1 comparisons.
Min and max together

- compare every two elements \( A[i], A[i+1] \)
- Compare larger against current max
- Smaller against current min

- \( 3\lfloor n / 2 \rfloor \)

Selection in expected O(n)

Randomised-select( A, p, r, i )

\[ \text{if } p = r \text{ then return } A[p] \]

\[ q = \text{Randomised-Partition}(A, p, r) \]

\[ k = q - p + 1 \quad \text{// nr of elements in subarr} \]

\[ \text{if } i <= k \]

\[ \text{then return } \text{Randomised-Partition}(A, p, q, i) \]

\[ \text{else return } \text{Randomised-Partition}(A, q+1, r, i-k) \]

Conclusion

- Sorting in general \( O(n \log n) \)
- Quicksort is rather good

- Linear time sorting is achievable when one does not assume only direct comparisons

- Find \( i \)’th value in unsorted – expected \( O(n) \)

- Find \( i \)’th value: worst case \( O(n) \) – see CLRS

Ok...

- lists – a versatile data structure for various purposes
- Sorting – a typical algorithm (many ways)
- Which sorting methods for array/list?

- Array: most of the important (e.g. update) tasks seem to be \( O(n) \), which is bad

Q: search for a value \( X \) in linked list?

A. \( O(1) \)

B. \( O(\log n) \)

C. \( O(n) \)

Can we search faster in linked lists?

- Why sort linked lists if search anyway \( O(n) \)?

- Linked lists:
  - what is the “mid-point” of any sublist ?
  - Therefore, binary search can not be used...

- Or can it?
Skip Lists

• A skip list for a set $S$ of distinct (key, element) items is a series of lists $S_0, S_1, \ldots, S_h$ such that
  - Each list $S_i$ contains the special keys $\infty$ and $-\infty$.
  - List $S_0$ contains the keys of $S$ in nondecreasing order.
  - Each list is a subsequence of the previous one, i.e., $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_h$.
  - List $S_h$ contains only the two special keys.

• We show how to use a skip list to implement the dictionary ADT.

typedef struct nodeStructure *node;
typedef struct nodeStructure{
  keyType key;
  valueType value;
  node forward[1]; /* variable sized array of forward pointers */
} node;
Skip Lists

Search
- We search for a key $x$ in a skip list as follows:
  - We start at the first position of the top list
  - At the current position $p$, we compare $x$ with $y = \text{key}(p)$
    - If $x = y$, we return $y$
    - If $x > y$, we "scan forward"
    - If $x < y$, we "drop down"
  - If we try to drop down past the bottom list, we return \texttt{NO_SUCH_KEY}
- Example: search for 78

• Example: search for 78

Randomized Algorithms
- A randomized algorithm performs coin tosses (i.e., uses random bits) to control its execution
- It contains statements of the type:
  - $b \leftarrow \text{random}()$,
  - \texttt{if} $b = 0$ \texttt{then} \texttt{A} \texttt{...}
  - \texttt{else} $b = 1$
- Its running time depends on the outcomes of the coin tosses
- We analyze the expected running time of a randomized algorithm under the following assumptions:
  - the coins are unbiased, and
  - the coin tosses are independent
- The worst-case running time of a randomized algorithm is often large but has very low probability (e.g., it occurs when all the coin tosses give "heads")
- We use a randomized algorithm to insert items into a skip list

Insertion
- To insert an item $(x, v)$ into a skip list, we use a randomized algorithm:
  - We repeatedly toss a coin until we get tails, and we denote with $i$ the number of times the coin came up heads
  - If $i > k$, we add to the skip list new lists $S_0, \ldots, S_{k-1}$, each containing only the two special keys
  - We search for $x$ in the skip list and find the positions $p_0, p_1, \ldots, p_{i}$ of the items with largest key less than $x$ in each list $S_0, S_1, \ldots, S_{i}$
    - For $j = 0, \ldots, i$, we insert item $(x, v)$ into list $S_j$ after position $p_j$
  - Example: insert key 15, with $i = 2$

Deletion
- To remove an item with key $x$ from a skip list, we proceed as follows:
  - We search for $x$ in the skip list and find the positions $p_0, p_1, \ldots, p_{i}$ of the items with key $x$, where position $p_{i}$ is in list $S_i$
    - We remove positions $p_0, p_1, \ldots, p_i$ from the lists $S_0, S_1, \ldots, S_{i}$
    - We remove all but one list containing only the two special keys
  - Example: remove key 34

Implementation v2
- We can implement a skip list with \textit{quad}-nodes
  - \textit{quad}-node stores:
    - $v$ item
    - Link to the node before
    - Link to the node after
    - Link to the node below
    - Link to the node after
  - Also, we define special keys \texttt{PLUS_\texttt{INF}} and \texttt{MINUS_\texttt{INF}}, and we modify the key comparator to handle them

Space Usage
- The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm
  - We use the following two basic probabilistic facts:
    - Fact 1: The probability of getting $i$ consecutive heads when flipping a coin is $\frac{1}{2^i}$
    - Fact 2: If each of $n$ items is present in a set with probability $p$, the expected size of the set is $n p$
  - The expected number of nodes used by the skip list is
    $$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} < 2n$$
  - Thus, the expected space usage of a skip list with $n$ items is $O(n)$
The running time of the search and insertion algorithms is affected by the height $h$ of the skip list. We show that with high probability, a skip list with $n$ items has height $O(\log n)$. We use the following additional probabilistic fact:

Fact 3: If each of $n$ events has probability $p$, the probability that at least one event occurs is at most $\sum_{i=1}^{n} p^i = \frac{p}{1-p}$.

Consider a skip list with $n$ items:
- By Fact 1, we insert an item in list $S_i$ with probability $\frac{1}{2^i}$.
- By Fact 3, the probability that list $S_i$ has at least one item is at most $\frac{1}{2^i}$.
- Thus a skip list with $n$ items has height at most $\log_2 n$ with high probability.

By picking $i = \log_2 n$, we have that the probability that list $S_i$ has at least one item is at most $\frac{1}{2^{\log_2 n}} = \frac{1}{n}$.

Thus a skip list with $n$ items has height at most $3\log \log n$ with probability at least $1 - \frac{1}{n}$.

The search time in a skip list is proportional to the number of drop-down steps, plus the number of scan-forward steps. The drop-down steps are bounded by the height of the skip list and thus are $O(\log n)$ with high probability. To analyze the scan-forward steps, we use yet another probabilistic fact:

Fact 4: The expected number of coin tosses required in order to get tails is 2.

When we scan forward in a list, the destination key does not belong to a higher list. A scan-forward step is associated with a former coin toss that gave tails. By Fact 4, in each list the expected number of scan-forward steps is 2. Thus, the expected number of scan-forward steps is $O(\log n)$.

We conclude that the expected space used is $O(n)$, the expected search, insertion and deletion time is $O(\log n)$.

A skip list is a data structure for dictionaries that uses a randomized insertion algorithm. In a skip list with $n$ items:
- The expected space used is $O(n)$.
- The expected search, insertion and deletion time is $O(\log n)$.
- Using a more complex probabilistic analysis, one can show that these performance bounds also hold with high probability.
- Skip lists are fast and simple to implement in practice.

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Linear data structures provide good versatility.

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Sorting in $O(n \log n)$ – Merge Sort, Quicksort.

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