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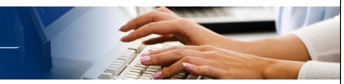

Advanced Algorithmics (6EAP)

MTAT.03.238

Succinct Trees

Jaak Vilo

Thanks to S. Srinivasa Rao

2013 Spring

Succinct Representations of Trees

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Seoul National University

Outline

- Succinct data structures
  - Introduction
  - Examples
- Tree representations
  - Motivation
  - Heap-like representation
  - Jacobson' s representation
  - Parenthesis representation
  - Partitioning method
  - Comparison and Applications
- Rank and Select on bit vectors

Succinct data structures

- Goal: represent the data in close to optimal space, while supporting the operations efficiently.

(optimal –– information-theoretic lower bound)

Introduced by [Jacobson, FOCS '89]

- An "extension" of data compression.

(Data compression:
  - Achieve close to optimal space
  - Queries need not be supported efficiently)

Applications

- Potential applications where
  - memory is limited: small memory devices like PDAs, mobile phones etc.
  - massive amounts of data: DNA sequences, geographical/astronomical data, search engines etc.

Examples

- Trees, Graphs
- Bit vectors, Sets
- Dynamic arrays
- Text indexes
  - suffix trees/suffix arrays etc.
- Permutations, Functions
- XML documents, File systems (labeled, multi-labeled trees)
- DAGs and BDDs
- …

Example: Text Indexing

- A text string T of length n over an alphabet Σ can be represented using
  - $n \log |\Sigma| + o(n \log |\Sigma|)$ bits,

  (or the even the k-th order entropy of T)

  to support the following pattern matching queries (given a pattern P of length m):
  - count the # occurrences of P in T,
  - report all the occurrences of P in T,
  - output a substring of T of given length

  in almost optimal time.

Example: Compressed Suffix Trees

- Given a text string T of length n over an alphabet Σ, one store it using $O(n \log |\Sigma|)$ bits, to support all the operations supported by a standard suffix tree such as pattern matching queries, suffix links, string depths, lowest common ancestors etc. with slight slowdown.
- Note that standard suffix trees use $O(n \log n)$ bits.

Example: Permutations

A permutation π of 1,…,n

	1	2	3	4	5	6	7	8
π:	6	5	2	8	1	3	4	7

A simple representation:
- n lg n bits
  - $\pi(i)$ in $O(1)$ time
  - $\pi^{-1}(i)$ in $O(n)$ time

$\pi(1)=6$ $\pi^{-1}(1)=5$

$\pi^2(1)=3$ $\pi^{-2}(1)=5$

…

Succinct representation:
- $(1+\varepsilon)\, n \lg n$ bits
  - $\pi(i)$ in $O(1)$ time
  - $\pi^{-1}(i)$ in $O(1/\varepsilon)$ time (`optimal' trade-off)
  - $\pi^k(i)$ in $O(1/\varepsilon)$ time (for any positive or negative integer k)
  - $\lg(n!) + o(n)$ (< n lg n) bits (optimal space)
  - $\pi^k(i)$ in $O(\lg n / \lg \lg n)$ time

Memory model

- Word RAM model with word size $\Theta(\log n)$ supporting
  - read/write
  - addition, subtraction, multiplication, division
  - left/right shifts
  - AND, OR, XOR, NOT

  operations on words in constant time.

(n is the "problem size")

Succinct Tree Representations

Motivation

Trees are used to represent:

- Directories (Unix, all the rest)
- Search trees (B-trees, binary search trees, digital trees or tries)
- Graph structures (we do a tree based search)
- Search indexes for text (including DNA)
  - Suffix trees
- XML documents
- …

Drawbacks of standard representations

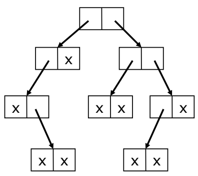
- Standard representations of trees support very few operations. To support other useful queries, they require a large amount of extra space.
- In various applications, one would like to support operations like "subtree size" of a node, "least common ancestor" of two nodes, "height", "depth" of a node, "ancestor" of a node at a given level etc.

Drawbacks of standard representations

- The space used by the tree structure could be the dominating factor in some applications.
  - Eg. More than half of the space used by a standard suffix tree representation is used to store the tree structure.
- "A pointer-based implementation of a suffix tree requires more than 20n bytes. A more sophisticated solution uses at least 12n bytes in the worst case, and about 8n bytes in the average. For example, a suffix tree built upon 700Mb of DNA sequences may take 40Gb of space."
  -- Handbook of Computational Molecular Biology, 2006

Standard representation

Binary tree:
each node has two pointers to its left and right children



An n-node tree takes 2n pointers or 2n lg n bits (can be easily reduced to n lg n + O(n) bits).

Supports finding left child or right child of a node (in constant time).

For each extra operation (eg. parent, subtree size) we have to pay, roughly, an additional n lg n bits.

Can we improve the space bound?

- There are less than 2^{2n} distinct binary trees on n nodes.
- 2n bits are enough to distinguish between any two different binary trees.
- Can we represent an n node binary tree using 2n bits?

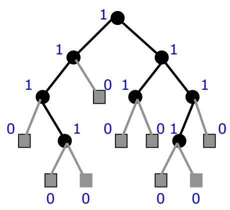
Heap-like notation for a binary tree

Add external nodes

Label internal nodes with a 1 and external nodes with a 0

Write the labels in level order

1 1 1 1 0 1 1 0 1 0 0 1 0 0 0 0 0



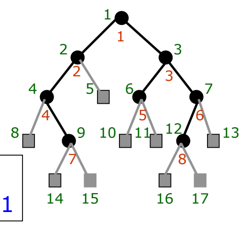
One can reconstruct the tree from this sequence

An n node binary tree can be represented in 2n+1 bits.

What about the operations?

Heap-like notation for a binary tree

left child(x) = [2x]
right child(x) = [2x+1]
parent(x) = [⌊x/2⌋]

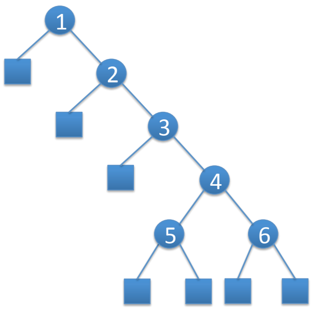


x → x: # 1's up to x
x → x: position of x-th 1

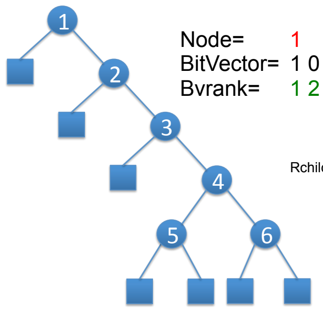
1	2	3	4		5	6		7			8					
1	1	1	1	0	1	1	0	1	0	0	1	0	0	0	0	0
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

COLOR

Example 2 (JV)



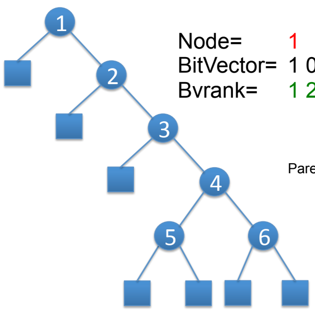
Example 2 (JV)



Node= 1 2 3 4 5 6
BitVector= 1 0 1 0 1 0 1 1 1 0 0 0 0
Bvrank= 1 2 3 4 5 6 7 8 9 10 11 12 13

Rchild(4) = Rank $_1$ (2*4 + 1) = 6

Example 2 (JV)



Node= 1 2 3 4 5 6
BitVector= 1 0 1 0 1 0 1 1 1 0 0 0 0
Bvrank= 1 2 3 4 5 6 7 8 9 10 11 12 13

Parent(5) =

5 => 8 => 4th node

4th node is at index 7

Rank/Select on a bit vector

Given a bit vector B

$rank_1(i)$ = # 1's up to position i in B

$select_1(i)$ = position of the i-th 1 in B
(similarly $rank_0$ and $select_0$)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
B:	0	1	1	0	1	0	0	0	1	1	0	1	1	1	1

Given a bit vector of length n, by storing an additional o(n)-bit structure, we can support all four operations in O(1) time.

$rank_1(5) = 3$
$select_1(4) = 9$
$rank_0(5) = 2$
$select_0(4) = 7$

An important substructure in most succinct data structures.

Implementations: [Kim et al.], [Gonzalez et al.], ...

Binary tree representation

- A binary tree on n nodes can be represented using 2n+o(n) bits to support:
  - parent
  - left child
  - right child

  in constant time.

[Jacobson '89]

Rank and Select on bit vectors

Rank/Select on a bit vector

Given a bit vector B

$rank_1(i)$ = # 1's up to position i in B

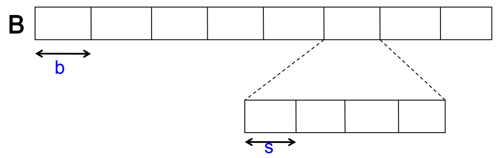
$select_1(i)$ = position of the i-th 1 in B

(similarly $rank_0$ and $select_0$)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
B:	0	1	1	0	1	0	0	0	1	1	0	1	1	1	1

$rank_1(5) = 3$
$select_1(4) = 9$
$rank_0(5) = 2$
$select_0(4) = 7$

Supporting Rank



- Store the rank up to the beginning of each block: (m/b) log m bits
- Store the `rank within the block' up to the beginning of each sub-block: (m/b)(b/s) log b bits
- Store a pre-computed table to find the rank within each sub-block: 2^s s log s bits

Rank/Select on bit vector

- Choosing b = $(\log m)^2$, and s = (1/2)log n makes the overall space to be O(m loglog m / log m) (= o(m)) bits.
- Supports rank in constant time.
- Select can also be supported in constant time using an auxiliary structure of size O(m loglog m / log m) bits.

[Clark-Munro '96] [Raman et al. '01]

Lower bounds for rank and select

- If the bit vector is read-only, any index (auxiliary structure) that supports rank or select in constant time (in fact in O(log m) bit probes) has size Ω(m loglog m / log m)

[Miltersen '05] [Golynski '06]

Space measures

- Bit-vector (BV):
  - space used be $m + o(m)$ bits.
- Bit-vector *index* :
  - bit-sequence stored in read-only memory
  - *index* of $o(m)$ bits to assist operations
- Compressed bit-vector: with n 1's
  - space used should be $B(m,n) + o(m)$ bits.

$$B(m,n) = \left\lceil \log \binom{m}{n} \right\rceil$$

Results on Bitvectors

- Elias (JACM 74)
- Jacobson (FOCS 89)
- Clark+Munro (SODA 96)
- Pagh (SICOMP 01)
- Raman et al (SODA 02)
- Miltersen (SODA 04)
- Golynski (ICALP 06)
- Gupta et al.

Implementations:

- Geary et al. (TCS 06)
- Kim et al. (WEA 05)
- Delpratt et al. (WEA 06, SOFSEM 07)
- Okanohara+Sadakane (ALENEX 07)

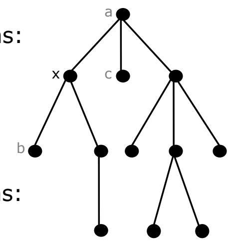
(Entry in *Encyclopaedia of Algorithms*)

Ordered trees

A rooted ordered tree (on n nodes):

Navigational operations:
- parent(x) = a
- first child(x) = b
- next sibling(x) = c

Other useful operations:
- degree(x) = 2
- subtree size(x) = 4



Ordered trees

- A binary tree representation taking 2n+o(n) bits that supports parent, left child and right child operations in constant time.
- There is a one-to-one correspondence between binary trees (on n nodes) and rooted ordered trees (on n+1 nodes).
- Gives an ordered tree representation taking 2n+o(n) bits that supports first child, next sibling (but not parent) operations in constant time.
- We will now consider ordered tree representations that support more operations.

Level-order degree sequence

Write the degree sequence in level order

3 2 0 3 0 1 0 2 0 0 0 0

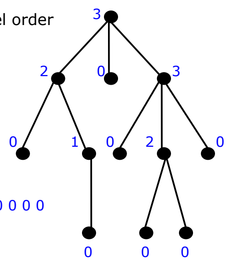
But, this still requires n lg n bits

Solution: write them in unary

1 1 1 0 1 1 0 0 1 1 1 0 0 1 0 0 1 1 0 0 0 0 0

Takes 2n-1 bits

A tree is uniquely determined by its degree sequence



Supporting operations

Add a dummy root so that each node has a corresponding 1

1 0 1 1 1 0 1 1 0 0 1 1 1 0 0 1 0 0 1 1 0 0 0 0 0
1 2 3 4 5 6 7 8 9 10 11 12

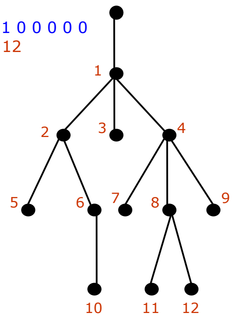
node k corresponds to the k-th 1 in the bit sequence

parent(k) = # 0's up to the k-th 1

children of k are stored after the k-th 0

supports: parent, i-th child, degree

(using rank and select)



Level-order unary degree sequence

- Space: 2n+o(n) bits
- Supports
  - parent
  - i-th child (and hence first child)
  - next sibling
  - degree

  in constant time.

Does not support subtree size operation.

[Jacobson '89]
[Implementation: Delpratt-Rahman-Raman '06]

Another approach

Write the degree sequence in depth-first order

3 2 0 1 0 0 3 0 2 0 0 0

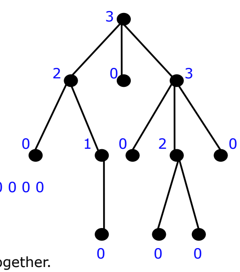
In unary:

1 1 1 0 1 1 0 0 1 0 0 0 1 1 1 0 0 1 1 0 0 0 0

Takes 2n-1 bits.

The representation of a subtree is together.

Supports subtree size along with other operations.
(Apart from rank/select, we need some additional operations.)



Depth-first unary degree sequence
(DFUDS)

- Space: 2n+o(n) bits
- Supports
  - parent
  - i-th child (and hence first child)
  - next sibling
  - degree
  - subtree size

  in constant time.

[Benoit et al. '05] [Jansson et al. '07]

Other useful operations

XML based applications:

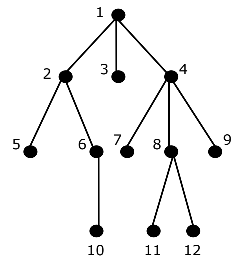
level ancestor(x,l): returns the ancestor of x at level l

eg. level ancestor(11,2) = 4

Suffix tree based applications:

LCA(x,y): returns the least common ancestor of x and y

eg. LCA(7,12) = 4



Parenthesis representation

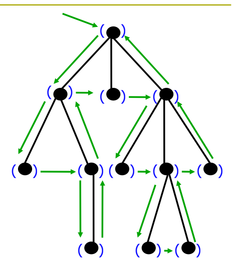
Associate an open-close parenthesis-pair with each node

Visit the nodes in pre-order, writing the parentheses

length: 2n

space: 2n bits

One can reconstruct the tree from this sequence



((( ) (( ))) ( ) (( ) (( ) ( )) ( )))

Operations

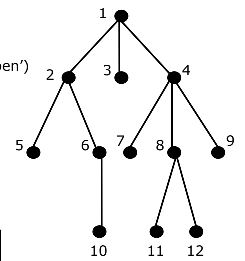
parent – enclosing parenthesis

first child – next parenthesis (if 'open')

next sibling – open parenthesis following the matching closing parenthesis (if exists)

subtree size – half the number of parentheses between the pair

with o(n) extra bits, all these can be supported in constant time



((( ) (( ))) ( ) (( ) (( ) ( )) ( )))

1 2 5 6 10 6 2 3 4 7 8 11 12 8 9 4 1

Parenthesis representation

- Space: 2n+o(n) bits
- Supports:

•parent	•level ancestor
•first child	•LCA
•next sibling	•leftmost/rightmost leaf
•subtree size	•number of leaves in the subtree
•degree	•next node in the level
•depth	•pre/post order number
•height	•i-th child

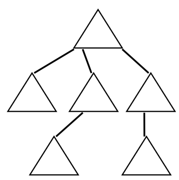
in constant time.

[Munro-Raman '97] [Munro et al. 01] [Sadakane '03] [Lu-Yeh '08]
[Implementation: Geary et al., CPM-04]

A different approach

- If we group k nodes into a block, then pointers with the block can be stored using only lg k bits.
- For example, if we can partition the tree into n/k blocks, each of size k, then we can store it using (n/k) lg n + (n/k) k lg k = (n/k) lg n +n lg k bits.

A careful two-level 'tree covering' method achieves a space bound of 2n+o(n) bits.



Tree covering method

- Space: 2n+o(n) bits
- Supports:

•parent	•level ancestor
•first child	•LCA
•next sibling	•leftmost/rightmost leaf
•subtree size	•number of leaves in the subtree
•degree	•next node in the level
•depth	•pre/post order number
•height	•i-th child

in constant time.

[Geary et al. ‘04] [He et al. ‘07] [Farzan-Munro ‘08]

Ordered tree representations

	parent, first child, sibling	i-th child, child rank	subtree size	degree	Depth, LCA	next node in the level	level ancestor	level-order rank, select	pre-order rank, select	post-order rank, select	DFUDS-order rank, select	leaf operations	height
LOUDS			X		X		X		X	X	X	X	X
DFUDS						X		X					X
PAREN								X			X		
PARTITION								X					

Unified representation

- A single representation that can *emulate* all other representations.
- Result: A 2n+o(n) bit representation that can *generate* an arbitrary word (O(log n) bits) of DFUDS, PAREN or PARTITION in constant time
- Supports the union of all the operations supported by each of these three representations.

[Farzan et al. ‘09]

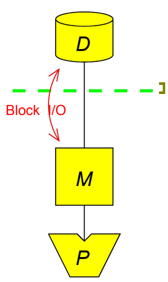
Applications

- Representing
  - suffix trees
  - XML documents (supporting XPath queries)
  - file systems (searching and Path queries)
  - representing BDDs
  - …

Open problems

- Making the structures dynamic (there are some existing results)
- Labeled trees (two different approaches supporting different sets of operations)
- Other memory models
  - External memory model (a few recent results)
  - Flash memory model
  - (So far mostly RAM model)

I/O Model [AV88]



- Parameters
  - N: Elements in structure
  - B: Elements per block
  - M: Elements in main memory

References

- Jacobson, FOCS 89
- Munro-Raman-Rao, FSTTCS 98 (JAlg 01)
- Benoit et al., WADS 99 (Algorithmica 05)
- Lu et al., SODA 01
- Sadakane, ISSAC 01
- Geary-Raman-Raman, SODA 04
- Munro-Rao, ICALP 04
- Jansson-Sadakane, SODA 06

Implementation:

- Geary et al., CPM 04
- Kim et al., WEA 05
- Gonzalez et al., WEA 05
- Delpratt-Rahman-Raman., WAE 06

Thank You