MTAT.05.125 Introduction to Theoretical Computer Science

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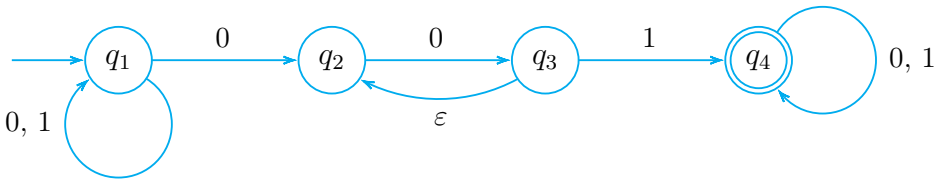
Lecture 6. Nondeterministic automata.

Nondeterminism

So far, the next state was uniquely defined, given the previous state and the input symbol. This is called *deterministic* computation. In a *nondeterministic* automaton several choices may exist for the next state at any point of computation.
Deterministic finite automaton: DFA.
Nondeterministic finite automaton: NFA.

Example 1. Nondeterministic automaton that accepts all strings containing 001:



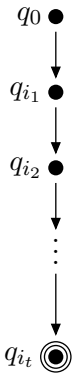
- Two outgoing edges 0 for q_1.
- No outgoing edge 1 for q_2.

Computation in NFA

- If there are several choices, the automaton runs several computations in parallel.

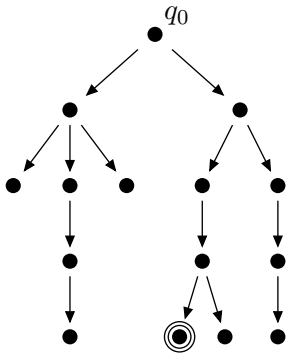
- If there is no particular symbol on output edges, the automaton discontinues the computation.
- If there is an outgoing edge labeled "ε", the the automaton runs several computations in parallel, starting with the current state and also the states pointed by the ε-arrow.
- NFA accepts the input string if and only if one of the parallel computations has accepted.

Deterministic



accepts

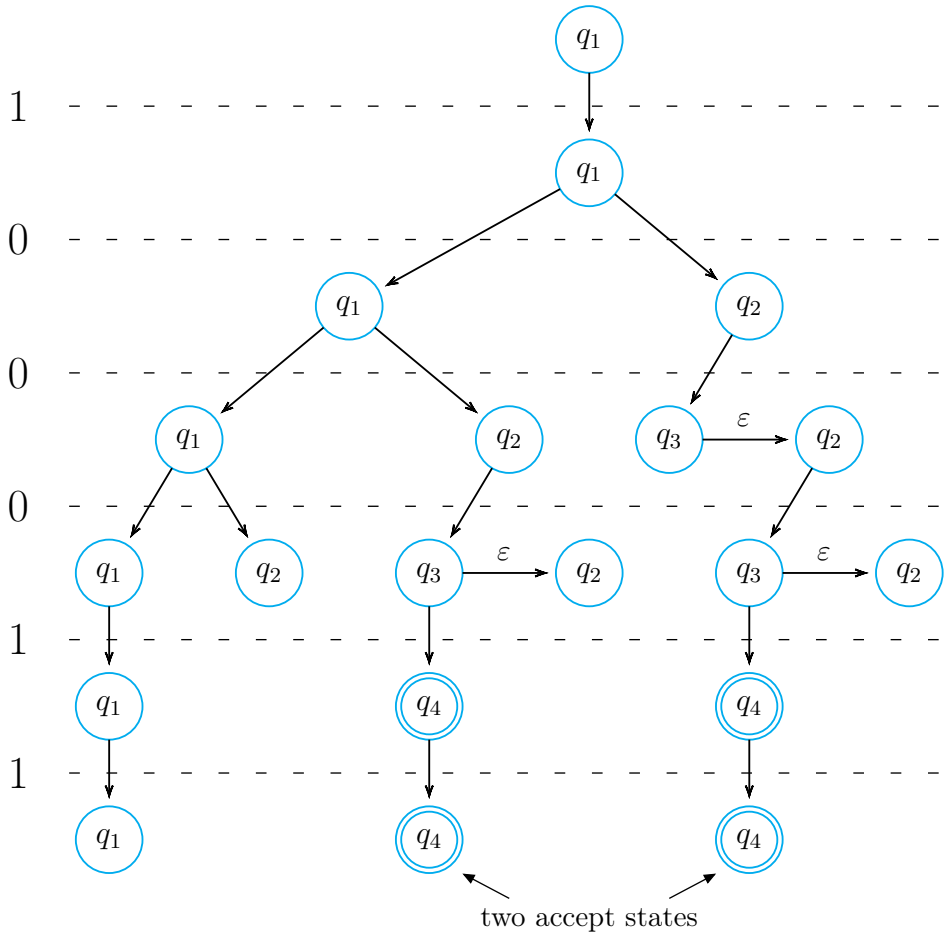
Nondeterministic



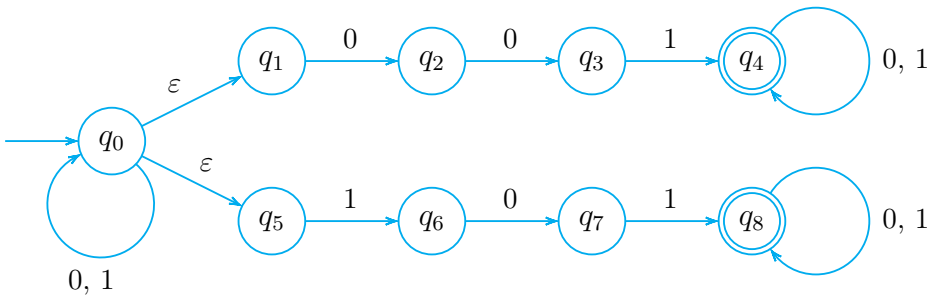
accepts

Automaton accepts input string, if there is an accept computation path.

Example 2. Consider how automaton from Example 1 works on input 100011. Computations form a tree.



Example 3. NFA that accepts all strings containing either 001 or 101 (or both).



Definition. *A* nondeterministic finite automaton *is a 5-tuple* $(Q, \Sigma, \delta, q_0, F)$, *where*

- Q *is a finite set of states;*
- Σ *is a finite alphabet;*

- $\delta\colon Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$ *is a transition function;*
- $q_0 \in Q$ *is a start state;*
- $F \subseteq Q$ *is a set of accept states.*

$\mathcal{P}(Q)$ is a set of all subsets of Q (called a *power set* of Q).
$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$.

Example 4. Let us apply the definition for the automaton from Example 1.

- Set of states $Q = \{q_1, q_2, q_3, q_4\}$.
- Alphabet $\Sigma = \{0, 1\}$.
- δ is described in the table:

	0	1	ε
q_1	$\{q_1, q_2\}$	$\{q_1\}$	$\varnothing$
q_2	$\{q_3\}$	$\varnothing$	$\varnothing$
q_3	$\varnothing$	$\{q_4\}$	$\{q_2\}$
q_4	$\{q_4\}$	$\{q_4\}$	$\varnothing$

- Start state is q_1.
- Set of accept states $F = \{q_4\}$.

Definition. *Let $M = (Q, \Sigma, \delta, q_1, F)$ be an NFA and w be a string over alphabet Σ. Then we say that M* accepts*the string w, if we can write $w = a_1 a_2 \dots a_m$, where $a_i \in \Sigma_\varepsilon$ (some $a_i = \varepsilon$, i.e. "empty"), and there is a sequence of states r_0, r_1, …, r_m, satisfying the following:*

1. *$r_0 = q_0$;*
2. *for each $i = 0, 1, \dots, m-1$ it holds $r_{i+1} \in \delta(r_i, a_{i+1})$;*
3. *$r_m \in F$.*

Definition. *Two automata are equivalent if they recognise the same language.*

Trivially, every DFA is an NFA. The opposite state can be formulated.

Theorem. *Every NFA has an equivalent DFA.*

Proof idea. Given NFA, we construct DFA that recognises the same language. If NFA has k states then the corresponding DFA will have 2^k states, representing all the subsets of states of NFA.

Proof. Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA that recognises a language L. We want to construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$, that recognises the same language L.

1) First consider a simple special case, where N does not have ε-arrows.

- $Q' = \mathcal{P}(Q)$, a set of all subsets of Q.
- For $R \in Q'$ and $a \in \Sigma$, let

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}. \qquad (*)$$

  In other words, if R is a state of M, it is also a set of states of N. When M reads a symbol a in state R, it can go to any of the states $\delta(r, a)$ for all $r \in R$.

- Start state: $q_0' = \{q_0\}$. M starts in the single state q_0.
- Accept states:

$$F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}.$$

2) Now consider ε arrows. Denote (so called ε-closure):

$E(R) = \{q \mid q$ can be reached from R by travelling from R by using ε arrows only$\}$.

(For instance, in Example 1 $E(\{q_1\}) = \{q_1\}$, $E(\{q_3\}) = \{q_2, q_3\}$). Replace (*) by

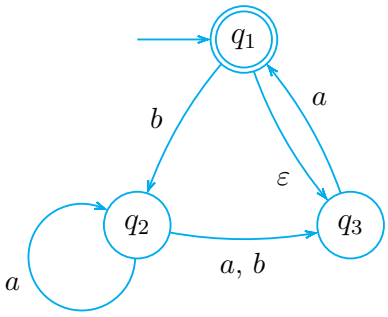
$$\delta'(R, a) = \bigcup_{r \in R} E(\delta(r, a)) = \{q \in Q \mid q \in E(\delta(r, a)) \text{ for some } r \in R\}.$$

We also need to replace the start state $\{q_0\}$ by $E(\{q_0\})$. This completes the construction.

The construction of M works correctly. At every step in the computation of M on an input, it enters a state that corresponds to a subset of states that N could be in at that point. □

Practise session

1. There is an NFA:



More formally, $N = (\{q_1, q_2, q_3\}, \{a, b\}, \delta, q_1, \{q_1\})$, where transition function is in the table

	a	b	ε
q_1	$\varnothing$	$\{q_2\}$	$\{q_3\}$
q_2	$\{q_2, q_3\}$	$\{q_3\}$	$\varnothing$
q_3	$\{q_1\}$	$\varnothing$	$\varnothing$

Construct an equivalent DFA.

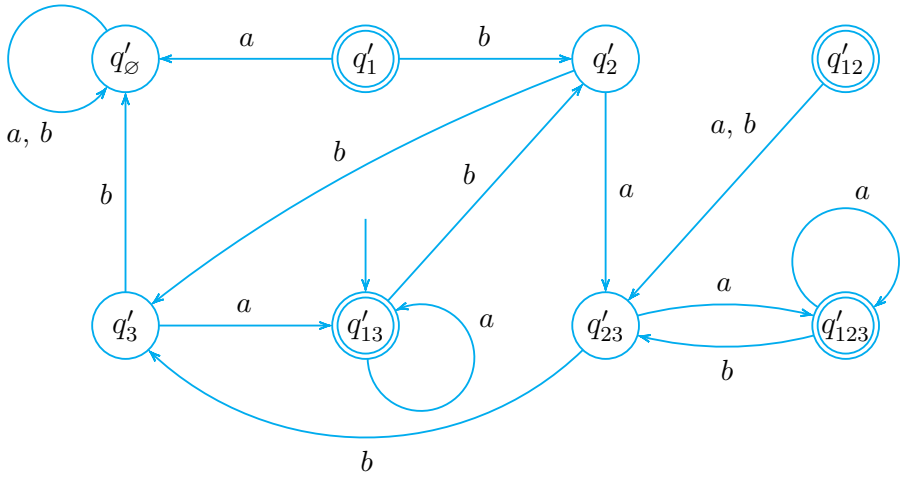
Solution. We construct $M = (Q', \{a, b\}, \delta', q'_0, F')$.

- $Q' = \{q'_\varnothing, q'_1, q'_2, q'_3, q'_{12}, q'_{13}, q'_{23}, q'_{123}\}$
- Transition function δ':

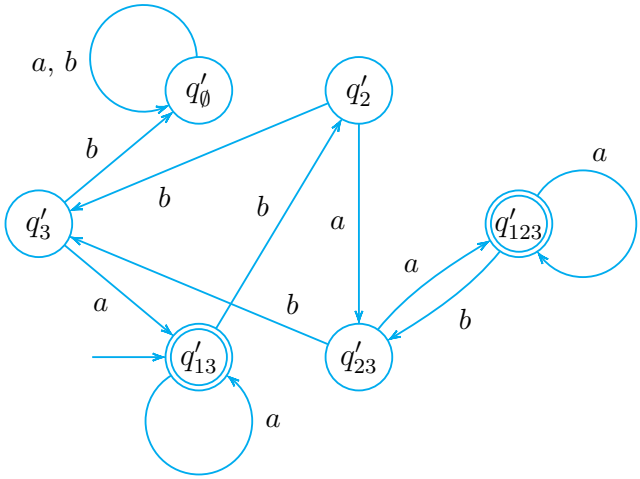
	a	b
$q'_\varnothing$	$q'_\varnothing$	$q'_\varnothing$
q'_1	$q'_\varnothing$	q'_2
q'_2	q'_{23}	q'_3
q'_3	q'_{13}	$q'_\varnothing$
q'_{12}	q'_{23}	q'_{23}
q'_{13}	q'_{13}	q'_2
q'_{23}	q'_{123}	q'_3
q'_{123}	q'_{123}	q'_{23}

- The start state q'_{13} (since in N we can arrive to q_3 from q_1 by using ε-arrows).
- $F = \{q'_1, q'_{12}, q'_{13}, q'_{123}\}$.

The resulting M is:

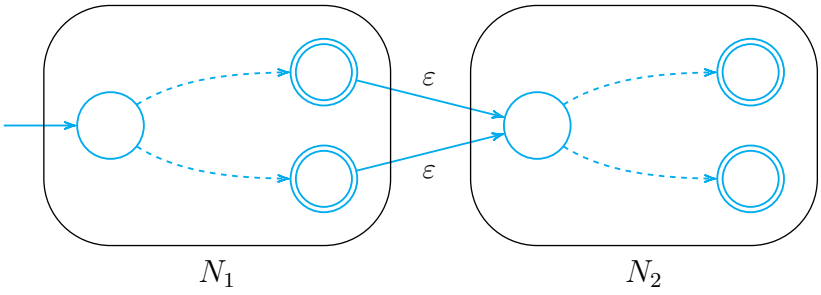


Note that we will never arrive to states q'_1 and q'_{12}. Therefore, they can be removed. We are left with the following DFA:



2. Let L_1 and L_2 be two regular languages. Prove that $L_1 \circ L_2$ (concatenation) is also a regular language.

Proof idea. Let N_1 and N_2 be two NFAs, recognising languages L_1 and L_2, respectively. We construct NFA N by connecting all accept states of N_1 to N_2 start state by ε-arrows:

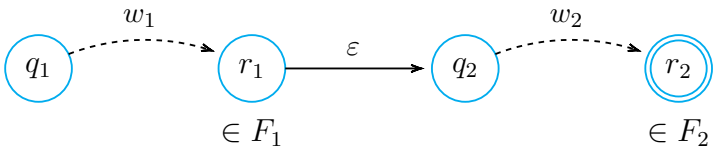


Solution. Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be NFAs recognising L_1 and L_2, respectively. We construct NFA $N = (Q, \Sigma, \delta, q_1, F_2)$ that recognises $L_1 \circ L_2$.

- States set $Q = Q_1 \cup Q_2$.
- Start state is q_1, i.e. the same as start state of N_1.
- Accept states set is F_2, i.e. the same as in N_2.
- Transition function δ is as follows:

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & \text{if } q \in Q_1,\ q \notin F_1, \\ \delta_1(q,a) & \text{if } q \in F_1,\ a \neq \varepsilon, \\ \delta_1(q,a) \cup \{q_2\} & \text{if } q \in F_1,\ a = \varepsilon, \\ \delta_2(q,a) & \text{if } q \in Q_2. \end{cases}$$

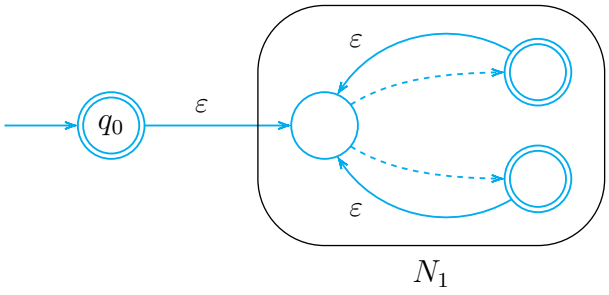
Automaton N accepts input words w if and only if there exists a computation path starting at q_1 and finishing in one of the states in F_2:



where $w = w_1 w_2$. This is equivalent to: $w_1 \in L_1$, $w_2 \in L_2$, $w = w_1 w_2$. And this is equivalent to $w \in L_1 \circ L_2$.

3. Prove that if L_1 is a regular language, then L_1^* is a regular language.

Proof idea. Let N_1 be an NFA recognising L_1. We construct a new automaton N so that each accept state of N_1 is connected to its old start state with ε-arrow. We also add the new start state, that is connected to an old start state with ε-arrow:

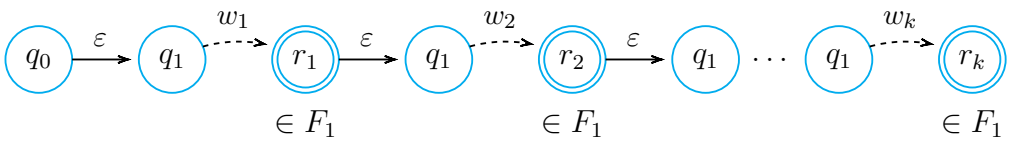


Solution. Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ be an NFA that recognises L_1. Construct an NFA $N = (Q, \Sigma, \delta, q_0, F)$, that recognises L_1^*.

- State set $Q = Q_1 \cup \{q_0\}$, i.e. we add new state q_0.
- New start state q_0.
- Set of accept states is $F = F_1 \cup \{q_0\}$.
- Transition function is

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & \text{if } q \in Q_1,\ q \notin F_1, \\ \delta_1(q,a) & \text{if } q \in F_1,\ a \neq \varepsilon, \\ \delta_1(q,a) \cup \{q_1\} & \text{if } q \in F_1,\ a = \varepsilon, \\ \{q_1\} & \text{if } q = q_0,\ a = \varepsilon, \\ \varnothing & \text{if } q = q_0,\ a \neq \varepsilon. \end{cases}$$

Suppose that N accepts input w. This is equivalent that either $w = \varepsilon$ or $w = w_1 w_2 \cdots w_k$, where:



This is equivalent that $w = \varepsilon$ or $w_1 \in L_1, w_2 \in L_1, \ldots, w_k \in L_1$ and $w = w_1 w_2 \cdots w_k$. This is equivalent that $w \in L_1^*$.