MTAT.05.125 Introduction to Theoretical Computer Science

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Lecture 4. Derangements.

Derangements

Example 1. *Hatchek Lady problem.* n gentlemen came to a theatre and gave their hats to a hatcheck lady. When they were leaving the theatre, every gentlemen got a hat which is not his (assume that the hats are all different). How many ways for doing so are there?

Definition. *Permuation of n elements $1, 2, \ldots, n$, such that no element i appears at place i, is called* derangement. *Number of derangements of n elements is sometimes denoted by $D(n)$.*

Solution. In this problem we want to count a number of derangements. If for $i = 1, 2, \ldots, n$ we define the property P_i as "gentleman i gets his hat", then derangements are such permutations of $1, 2, \ldots, n$ that satisfy *none* of the properties $P_1, P_2, \ldots, P_n$. We can apply inclusion-exclusion principle:

- $W(0)$ is the number of all permutations of n hats, hence $W(0) = n!$
- Now count the number of permutations of hats satisfying P_i, i.e. that gentleman i gets his hat. This is the same as if we give the hat i to gentleman i and have no restrictions on rest of the hats. Therefore, $W(P_i)$ equals to the number of ways to put numbers $1, 2, \ldots, i-1, i+1, \ldots, n$ ($n-1$ numbers) into positions $1, 2, \ldots, i-1, i+1, \ldots, n$ ($n-1$ positions). This is exactly the number of permutations of $n-1$ objects, therefore $W(P_i) = (n-1)!$ for each $i = 1, 2, \ldots, n$.

  That being so, $W(1) = W(P_1) + W(P_2) + \cdots + W(P_n) = n \cdot (n-1)!$

- To count the number of permutations of hats satisfying both P_i and P_j $(1 \leq i, j \leq n,\ i \neq j)$[1], we can give the hats i and j to the gentleman i and the gentleman j respectively and then count the number of ways to permute remaining $n-2$ hats. Thus $W(P_i, P_j) = (n-2)!$

  Now, to count $W(2)$ we sum up the values of $W(P_i, P_j)$ for all pairs $\{P_i, P_j\}$ without respect to order:

  $$W(2) = W(P_1, P_2) + W(P_1, P_3) + \cdots + W(P_1, P_n) \\ + W(P_2, P_3) + \cdots + W(P_{n-1}, P_n) = \binom{n}{2} \cdot (n-2)!$$

  This is so because there are $\binom{n}{2}$ ways to choose pair of properties out of n if we are not concerned about order in a pair.

- Analogously $W(P_i, P_j, P_k) = (n-3)!$ and $W(3) = W(P_1, P_2, P_3) + \ldots + W(P_{n-2}, P_{n-1}, P_n) = \binom{n}{3}(n-3)!$

- ...

- Finally, $W(P_1, P_2, \ldots, P_n) = 1$ (every gentleman gets his own hat) and $W(n) = 1$.

As a result,

$$\begin{aligned} D(n) = E(0) &= W(0) - W(1) + W(2) - W(3) + \ldots + (-1)^n W(n) = \\ &= n! - n \cdot (n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^n \binom{n}{n} = \\ &= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right). \end{aligned}$$

From Calculus course you might know, that

$$\lim_{n \to \infty} \frac{D(n)}{n!} = \frac{1}{\mathrm{e}} \approx 0{,}3679.$$

This means that when n is large, probability that none of the gentlemen gets his own hat is close to $1/\mathrm{e}$.

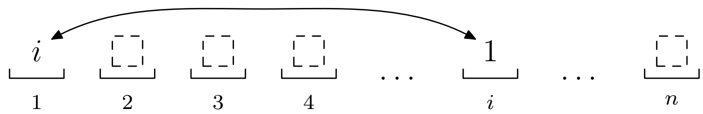
Example 2. Prove that $D(n) = (n-1)\big(D(n-1) + D(n-2)\big)$.

[1] Recall that we do not care about order of properties in brackets, i.e. $W(P_i, P_j)$ is the same as $W(P_j, P_i)$ and we consider a pair $\{P_i, P_j\}$ only once.

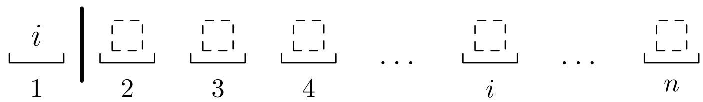
Solution. L.H.S. of the equation is $D(n)$, the number of ways to arrange elements $1, 2, \ldots, n$ in places $1, 2, \ldots, n$, such that for every i the element i is not in the place i.

R.H.S.: Consider place 1. Assume that element i $(2 \le i \le n)$ is located at place 1. Then there are two cases:

1. If element 1 is located at place i (i.e. 1 and i exchanged their locations), then there are $n-2$ elements left, which are arranged at positions $2, 3, \ldots, i-1, i+1, \ldots, n$ (and none of them is in its location); this can be done in $D(n-2)$ ways (derangements of elements $2, 3, \ldots, i-1, i+1, \ldots, n$).



2. If element 1 is in a different (from i) location, then rename "1" into "i". Now the elements $2, 3, \ldots,$ new $i, \ldots, n$ are all in the positions that differ from the corresponding number. This can be done in $D(n-1)$ ways.



So after we fixed element i in the first position, there are $D(n-1)+D(n-2)$ possibilities. Since i can obtain values from $2, 3, \ldots, n$, the total number of derangements of n is $(n-1)\big(D(n-2)+D(n-1)\big)$.

Thus we counted the number of derangements of n elements in two different ways. This number should be the same which proves the formula.

Practice session

1. How many integer solutions has the equation $x_1 + x_2 + x_3 = 25$, if $x_1 \le 5$, $x_2 \le 10$, $x_3 \le 18$ and x_1, x_2, $x_3 \ge 0$?

2. How many ways are there to order the digits 0, 1, 2, …, 9 (no repetitions allowed), such that the subsequences 012, 234 and 456 do not appear?

3. Give a combinatorial proof that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n = n!$$

Solutions to the practice session questions

1. How many integer solutions has the equation $x_1 + x_2 + x_3 = 25$, if $x_1 \le 5$, $x_2 \le 10$, $x_3 \le 18$ and x_1, x_2, $x_3 \ge 0$?

Solution. We want to apply inclusion-exclusion principle. The principle operates on elements possessing/not possessing some properties. By element here we will understand an individual integer solution of the following more general equation:

$$x_1 + x_2 + x_3 = 25\,, \qquad \text{with } x_1, x_2, x_3 \ge 0\,. \tag{1}$$

Define properties $P_1 =$ "$x_1 \ge 6$", $P_2 =$ "$x_2 \ge 11$" and $P_3 =$ "$x_3 \ge 19$". These are exactly negations of the requirements $x_1 \le 5$, $x_2 \le 10$, $x_3 \le 18$, respectively. Therefore our original problem can be re-formulated as finding the number of elements (=solutions of (1)) that possess *none* of the properties P_1, P_2, P_3.

We apply inclusion-exclusion principle.

- First of all, $W(0) = \left(\!\binom{3}{25}\!\right) = \binom{3+25-1}{25}$, because it is a number of solutions of the equation (1) (we do not take any of P_1, P_2, P_3 into consideration). This was given along with the definition of combinations with repetitions (see Lecture 2, Example 5).

- Consider $W(P_1)$, i.e. the number of solutions of (1) that has $x_1 \ge 6$. We can find the number of solutions in the following manner. We can define $x_1' = x_1 - 6$ (i.e. $x_1 = x_1' + 6$), and then (1) with requirement P_1 is equivalent to the following equation that have a "canonical"[2] form:

  $$x_1' + x_2 + x_3 = 25 - 6 = 19, \qquad x_1', x_2, x_3 \ge 0\,.$$

  Number of its solutions is $\left(\!\binom{3}{19}\!\right) = \binom{3+19-1}{19}$, i.e. $W(P_1) = \binom{3+19-1}{19}$.

  Arguing in the same manner, we get that $W(P_2) = \left(\!\binom{3}{14}\!\right) = \binom{3+14-1}{14}$ and $W(P_3) = \left(\!\binom{3}{6}\!\right) = \binom{3+6-1}{6}$. From that it follows $W(1) = \binom{21}{19} + \binom{16}{14} + \binom{8}{6}$.

- In order to find $W(P_1, P_2)$ we can use the same trick, but now we introduce two variables, $x_1' = x_1 - 6$ and $x_2' = x_2 - 11$ and then (1) with requirements P_1 and P_2 is equivalent to the following equation that have a "canonical" form:

  $$x_1' + x_2' + x_3 = 25 - 6 - 11 = 8, \qquad x_1', x_2', x_3 \ge 0\,.$$

[2] "Canonical" here means that equation has exactly the form as in Lecture 3, Example 5. This is not a standard term for such equations.

It has $\left(\!\binom{3}{8}\!\right) = \binom{3+8-1}{8}$ integer solutions and therefore $W(P_1, P_2) = \binom{3+8-1}{8}$.

Analogously, $W(P_1, P_3) = \left(\!\binom{3}{0}\!\right) = \binom{3+0-1}{0} = 1$ (obvious, because the only solution is $x_1 = 6, x_2 = 0, x_3 = 19$). And $W(P_2, P_3) = 0$ since there are no solutions with $x_2 \geq 11$ and $x_3 \geq 19$ (that would require $x_1 + x_2 + x_3 \geq 11 + 19 = 30$).

So $W(2) = \binom{10}{8} + 1 + 0$

- $W(P_1, P_2, P_3) = 0$ and thus $W(3) = 0$.

That being so, the number of solutions of original equation is

$$E(0) = W(0) - W(1) + W(2) = 351 - 358 + 46 = 39.$$

2. How many ways are there to order the digits 0, 1, 2, ..., 9 (no repetitions allowed), such that the subsequences 012, 234 and 456 do not appear?

Solution. Define following properties on permutations of digits 0, 1, 2, ..., 9: P_1 = "contains 012", P_2 = "contains 234" and P_3 = "contains 456".

- Without restrictions, the number of all permutations of digits 0, 1, 2, ..., 9 is 10!, thus $W(0) = 10!$.

- $W(P_1)$ is the number of permutations that do contain 012. We can use the technique we used before – consider 012 as one object, therefore we need to count number of permutations of 8 objects, i.e. 012, 3, 4, ..., 9. So $W(P_1) = 8!$. Analogously $W(P_2) = W(P_3) = 8!$. Hence $W(1) = 3 \cdot 8!$.

- $W(P_1, P_2)$ is the number of permutations that contain both 012 and 234. Since we have only one digit 2, fulfilling both P_1 and P_2 is the same as fulfilling the requirement "permutation contains 01234". Again, we consider 01234 as a single object and $W(P_1, P_2) = 6!$. Analogously $W(P_2, P_3) = 6!$.

  However, for counting $W(P_1, P_3)$, 012 and 456 do not have same digits, therefore we argue as follows. Let us consider 012 as a single object, and also consider 456 as a single object. Therefore we have now 012, 3, 456, 7, 8, 9 and all there are 6! permutations of these objects. So $W(P_1, P_3) = 6!$

  In total, $W(2) = 3 \cdot 6!$

- Finally, $W(3) = W(P_1, P_2, P_3) = 4!$, because we need here to count all permutations of objects 0123456, 7, 8, 9.

The answer:

$$E(0) = W(0) - W(1) + W(2) - W(3) = 10! - 3 \cdot 8! + 3 \cdot 6! - 4!.$$

3. Give a combinatorial proof that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n = n!$$

Solution. We will try to find a combinatorial problem, such that L.H.S. is an expression for $E(0)$ for that problem. R.H.S. suggests that number of permutations on n elements is a possible choice.

We want to define the properties so that $W(0) = n^n$ and $W(i) = \binom{n}{i}(n-i)^n$.

$W(0)$ is a number of all elements under consideration. It is easy to see, that number of vectors of length n over elements $\{1, 2, \ldots, n\}$ is exactly n^n. Obviously, all permutations of n elements is a subset of all such vectors with additional requirement that every number $1, 2, \ldots, n$ exactly once.

Let us define the properties P_i = "the element i does not appear in the vector" for $i = 1, 2, \ldots, n$. With this definition the vectors that satisfy *none* of these properties are exactly the permutations. Then we apply inclusion-exclusion principle.

- $W(0) = n^n$.
- For $1 \leq i \leq n$: $W(P_i) = (n-1)^n$ and thus $W(1) = \binom{n}{1}(n-1)^n$.
- For $1 \leq i, j \leq n, i \neq j$: $W(P_i, P_j) = (n-2)^n$ and thus $W(2) = \binom{n}{2}(n-2)^n$.
- More generally, for any r properties, $W(P_{i_1}, P_{i_2}, \ldots, P_{i_r}) = (n-r)^n$ and therefore $W(r) = \binom{n}{r}(n-r)^n$.

Therefore

$$E(0) = \sum_{i=0}^{n} (-1)^i W(i) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n ,$$

which proves the equation.