

# MTAT.05.125 Introduction to Theoretical Computer Science

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Lecture 3. Newton Binomial. Inclusion-exclusion principle.

## Newton Binomial

Example 1. We know the equalities:

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = (x + y)(x^2 + 2xy + y^2) = x^3 + 3x^2y + 3xy^2 + y^3$$

What is  $(x + y)^n$  then? The answer is given by the following theorem.

**Theorem.** (*Newton binomial.*)

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}.$$

*Proof.* Write

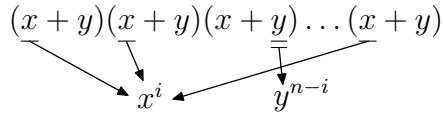
$$(x + y)^n = \underbrace{(x + y)(x + y) \cdot \dots \cdot (x + y)}_{n \text{ times}}.$$

Expanding this is essentially the same as choosing  $x$  or  $y$  in every parenthesis and then summing up obtained terms<sup>1</sup>. After summation, how many times do we obtain the term  $x^i y^{n-i}$ ?

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<sup>1</sup>For example, if  $n = 3$  we have:  $(x + y)(x + y)(x + y) = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy = x^3 + 3x^2y + 3xy^2 + y^3$ .

- $x$  comes from  $i$  multipliers;
- $y$  comes from rest  $n - i$  multipliers.

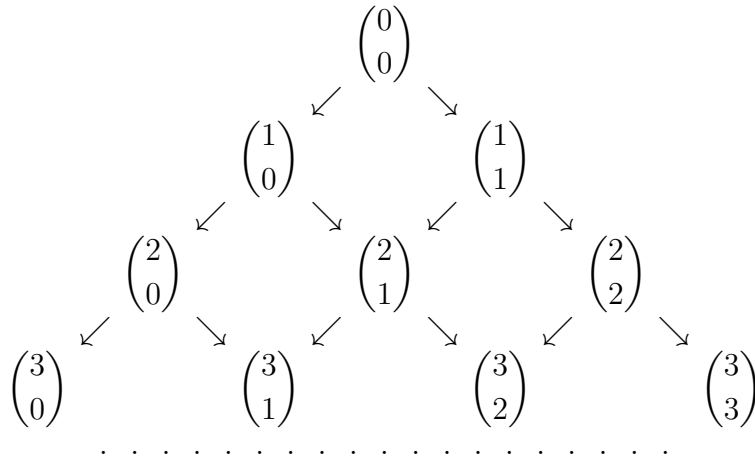


Number of ways to choose  $i$  multipliers out of  $n$  is  $\binom{n}{i}$ . □

We saw in the practice session that

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

If we denote  $\binom{n}{0} = 1$  for all  $n \geq 0$  and  $\binom{n}{k} = 0$  if  $k < 0$  or  $k > n$ , then we can arrange all the binomial coefficients in infinite triangle table:



Here every element is a sum of elements according to incoming arrows. Elements along the edges are all equal to 1. This table is called Pascal Triangle. We can write exact numbers:

|  |   |   |   |   |   |
|--|---|---|---|---|---|
|  |   |   | 1 |   |   |
|  |   | 1 |   | 1 |   |
|  | 1 | 2 | 1 |   |   |
|  | 1 | 3 | 3 | 1 |   |
|  | 1 | 4 | 6 | 4 | 1 |
|  |   |   |   |   |   |

The  $n$ 'th row in the triangle gives coefficients of  $(x + y)^n$ . For instance,  $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ .

Let us consider some important special cases. Assign  $y = 1$ , then we have:

$$(1 + x)^n = \sum_{i=0}^n \binom{n}{i} x^i.$$

Here assign  $x = 1$ :

$$\sum_{i=0}^n \binom{n}{i} = 2^n. \quad (*)$$

But if we assign  $x = -1$ , in the same manner we get:

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 0. \quad (**)$$

From (\*\*) we get

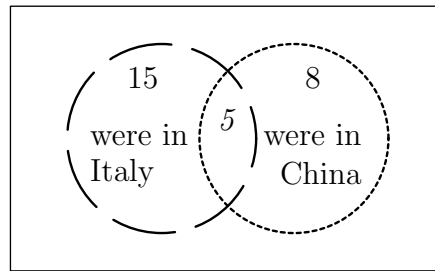
$$\sum_{\substack{0 \leq i \leq n, \\ \text{odd } i}} \binom{n}{i} = \sum_{\substack{0 \leq i \leq n, \\ \text{even } i}} \binom{n}{i}. \quad (**)$$

Comparing (\*) and (\*\*), we obtain:

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}.$$

## Inclusion-exclusion principle

**Example 2.** There are 20 students in the class. 15 of them were in Italy and 8 were in China. 5 students were in both countries. How many of the students were neither in Italy or China?



20 students

*Solution.* We see that some of the students visited at least one country. 15 students were in Italy and 8 students were in China. This gives us in total  $15 + 8 = 23$  students. However, we know that there are 5 students that visited both countries – we counted these students twice: once for Italy visit and once for China visit. We need to subtract them:

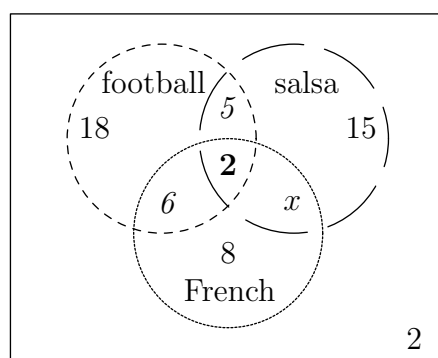
$$\underbrace{20}_{\text{all the students}} - \left( \underbrace{15 + 8 - 5}_{\text{were to at least one of the countries}} \right) = \underbrace{2}_{\text{were to neither of the countries}}$$

This is exactly the number of students who were to at least one country. Therefore the rest 2 students were to neither of the countries.

Example 3. There are 30 students in the class.

- 18 students play football. 15 students dance salsa. 8 students speak French.
- 5 students play football and dance salsa.
- 6 students play football and speak French.
- 2 students play football, dance salsa and speak French.

How many students dance salsa and speak French, if 2 students don't do any of these three activities?



30 students

*Solution.* From the picture we see the sizes of the sets of students doing these activities. Therefore:

$$30 - (18 + 15 + 8) + (6 + 5 + x) - 2 = 2.$$

**Inclusion-exclusion principle.** We have  $n$  elements, possessing some of the properties  $P_1, P_2, \dots, P_t$ . Denote

- $W(P_i)$  number of elements with property  $P_i, 1 \leq i \leq t$
- $W(P_i, P_j)$  number of elements with properties  $P_i$  and  $P_j, 1 \leq i, j \leq t$
- $\dots$
- $W(P_1, P_2, \dots, P_t)$  number of elements with properties  $P_1, P_2, \dots, P_t$
- $E(0)$  number of elements without any of these properties.

Then

$$\begin{aligned}
 E(0) = & n - (W(P_1) + W(P_2) + \dots + W(P_t)) + \\
 & + (W(P_1, P_2) + W(P_1, P_3) + \dots + W(P_{t-1}, P_t)) - \\
 & - (W(P_1, P_2, P_3) + W(P_1, P_2, P_4) + \dots + W(P_{t-2}, P_{t-1}, P_t)) + \\
 & + \dots + (-1)^t W(P_1, P_2, \dots, P_t).
 \end{aligned}$$

*Proof.* Consider arbitrary element  $x$ . Assume that  $x$  has exactly  $j \geq 1$  properties. Without loss of generality assume that these properties are  $P_1, P_2, \dots, P_j$ .

The element  $x$  is counted by  $W(P_{i_1}, P_{i_2}, \dots, P_{i_k})$  if and only if a set  $\{P_1, P_2, \dots, P_j\}$  contains all of the properties  $P_{i_1}, P_{i_2}, \dots, P_{i_k}$ . There are  $\binom{j}{k}$  such choices of properties. The sign of the expression, when counting  $k$  tuples of properties, is  $(-1)^k$ . Therefore  $x$  contributes to the R.H.S. the quantity:

$$1 - j + \binom{j}{2} - \binom{j}{3} + \dots + (-1)^j \binom{j}{j}.$$

From (\*\*) the above expression is 0. Thus the contribution of  $x$  to both sides of the formula is 0. This is true for every  $x$  with  $j \geq 1$  properties.

Now,  $x$  with 0 properties contributes 1 to the L.H.S. and 1 to the R.H.S. As a result, contribution of every  $x$  to the L.H.S. and R.H.S. is equal, and hence the formula has been proven.  $\square$

If we introduce the following notation:

$$\begin{aligned}
 W(0) &= n \\
 W(1) &= W(P_1) + W(P_2) + \dots + W(P_t) \\
 W(2) &= W(P_1, P_2) + W(P_1, P_3) + \dots + W(P_{t-1}, P_t) \\
 W(3) &= W(P_1, P_2, P_3) + W(P_1, P_2, P_4) + \dots + W(P_{t-2}, P_{t-1}, P_t) \\
 &\dots \\
 W(t) &= W(P_1, P_2, \dots, P_t),
 \end{aligned}$$

then inclusion-exclusion principle can be written shorter:

$$E(0) = \sum_{i=0}^t (-1)^i W(i).$$

## Practise session

1. Prove

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}.$$

2. Prove that

$$\sum_{i=0}^n i \binom{n}{i} = n \cdot 2^{n-1}.$$

3. Prove that

$$\sum_{i=0}^n (i-2) \binom{n}{i} = 2^{n-1}(n-4).$$

4. There are 18 students in the class. 12 of them speak French, 7 speak Italian and 5 speak Japanese. 5 of them speak Italian and French. 2 speak Italian and Japanese. 1 speaks French and Japanese. And one student speak all three languages. How many students don't speak any of the three languages?
5. How many numbers between 1 and 100 are divisible neither by 2, or 3, or 7?

## Solutions to practise session questions

1. Prove

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}.$$

*Solution 1.* Since  $\binom{n}{i}^2 = \binom{n}{i} \binom{n}{i} = \binom{n}{i} \binom{n}{n-i}$ , our identity is equivalent to the following one:

$$\sum_{i=0}^n \binom{n}{i} \binom{n}{n-i} = \binom{2n}{n}.$$

Consider the following problem: we have  $n$  white balls numbered  $1, 2, \dots, n$  and  $n$  black balls numbered  $1, 2, \dots, n$ . How many ways are there to choose  $n$  balls out of these  $2n$  balls?

Let us count in two different ways.

(a) Choose  $n$  balls out of  $2n$  balls (all balls are different):

$$\binom{2n}{n}.$$

(b) Let  $i$  be the number of chosen white balls,  $0 \leq i \leq n$ . The number of ways to choose  $i$  white balls out of  $n$  (different) white balls:  $\binom{n}{i}$ . The number of ways to choose  $n-i$  black balls out of  $n$  (different) black balls:  $\binom{n}{n-i}$ . By the multiplication principle, the total number of choices of  $i$  white and  $n-i$  black balls:  $\binom{n}{i} \binom{n}{n-i}$ . Since any  $i$ ,  $0 \leq i \leq n$  is possible, we have to sum up:

$$\sum_{i=0}^n \binom{n}{i} \binom{n}{n-i}.$$

Comparing (a) and (b), we get required identity.

*Solution 2.* From Newton Binomial:

$$(x+y)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i y^{2n-i}.$$

$\binom{2n}{n}$  is a coefficient of  $x^n y^n$  in this binomial.

On the other hand,

$$(x+y)^{2n} = (x+y)^n (x+y)^n = \underbrace{\left( \sum_{i=0}^n \binom{n}{i} x^i y^{n-i} \right)}_{\text{take } x^i y^{n-i}} \underbrace{\left( \sum_{i=0}^n \binom{n}{i} x^i y^{n-i} \right)}_{\text{take } x^{n-i} y^i}.$$

$x^n y^n$  is obtained by multiplying  $x^i y^{n-i}$  from the first sum by  $x^{n-i} y^i$  from the second sum. The value of  $i$  could be anything between 0 and  $n$ . Hence,

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\text{coefficient} \\ \text{of } x^i \\ \text{in the left} \\ \text{multiplier}}} \underbrace{\binom{n}{n-i}}_{\substack{\text{coefficient} \\ \text{of } x^{n-i} \\ \text{in the right} \\ \text{multiplier}}} = \binom{2n}{n}.$$

2. Prove that

$$\sum_{i=0}^n i \binom{n}{i} = n \cdot 2^{n-1}.$$

*Solution.* From the Newton Binomial:

$$(1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i.$$

This is true for all  $x \in \mathbb{R}$ . Take a derivative of both sides:

$$n(1+x)^{n-1} = \sum_{i=1}^n i \binom{n}{i} x^{i-1}.$$

(The term corresponding to  $i = 0$  becomes 0.) Substitute  $x = 1$ :

$$n \cdot 2^{n-1} = \sum_{i=1}^n i \binom{n}{i}.$$

3. Prove that

$$\sum_{i=0}^n (i-2) \binom{n}{i} = 2^{n-1}(n-4).$$

*Solution.* We know already that

$$\sum_{i=0}^n \binom{n}{i} = 2^n.$$

From the previous problem:

$$\sum_{i=1}^n i \binom{n}{i} = n \cdot 2^{n-1}$$



Subtract them:

$$\sum_{i=1}^n i \binom{n}{i} - 2 \sum_{i=0}^n \binom{n}{i} = \sum_{i=0}^n i \binom{n}{i} - \sum_{i=0}^n 2 \binom{n}{i} = \sum_{i=0}^n (i-2) \binom{n}{i}.$$

On the other hand:

$$n \cdot 2^{n-1} - 2 \cdot 2^n = 2^{n-1}(n-4).$$

4. There are 18 students in the class. 12 of them speak French, 7 speak Italian and 5 speak Japanese. 5 of them speak Italian and French. 2 speak Italian and Japanese. 1 speaks French and Japanese. And one student speak all three languages. How many students don't speak any of the three languages?

*Solution.* Use inclusion-exclusion principle with the following properties considered:  $P_1$  = speaks French,  $P_2$  = speaks Italian,  $P_3$  = speaks Japanese. Let us compute:

- $W(0) = 18$
- $W(1) = W(P_1) + W(P_2) + W(P_3) = 12 + 7 + 5 = 24$
- $W(2) = W(P_1, P_2) + W(P_1, P_3) + W(P_2, P_3) = 5 + 1 + 2 = 8$
- $W(3) = 1$

Therefore

$$E(0) = W(0) - W(1) + W(2) - W(3) = 18 - 24 + 8 - 1 = 1.$$

5. How many numbers between 1 and 100 are divisible neither by 2, or 3, or 7?

*Solution.* Define properties:  $P_1$  = the number is divisible by 2,  $P_2$  = the number is divisible by 3,  $P_3$  = the number is divisible by 7. Then:

- $W(0) = 100.$
- $W(P_1) = 50, W(P_2) = 33, W(P_3) = 14.$  Thus  $W(1) = W(P_1) + W(P_2) + W(P_3) = 97.$
- $W(P_1, P_2) = \lfloor \frac{100}{6} \rfloor = 16, W(P_1, P_3) = 7, W(P_2, P_3) = 4.$  Thus  $W(2) = W(P_1, P_2) + W(P_1, P_3) + W(P_2, P_3) = 27.$
- $W(3) = W(P_1, P_2, P_3) = 2.$

Applying inclusion-exclusion principle:

$$E(0) = W(0) - W(1) + W(2) - W(3) = 100 - 97 + 27 - 2 = 28.$$

## Additional exercises

6. There is a grid of size  $n \times n$ . In lower left knot point there are  $2^n$  buttons. At the first move half of them move up and half move right. At the second move half of the buttons at both new knot points move up and half move right.  $n$  such moves are made. Where do the buttons lie after  $n$  moves and how many buttons are there in each knot point of the grid?

7. Using the expansions of the expression  $(1+x)^n$  and its derivative, prove the equality

$$\sum_{i=1}^n i \binom{n}{i}^2 = \frac{n}{2} \binom{2n}{n}.$$

8. Prove

$$\sum_{i=0}^n \binom{n+i}{n} = \binom{2n+1}{n+1}.$$

9. In how many ways can the letters A, B, ..., H be arranged, such that the resulting word does not contain letter combinations AB, CD, EF, and GH?
10. Six people come to visit and everybody hangs his hat on the rack in the vestibule. Later, when they prepare to leave, suddenly the light goes off and everybody takes one hat from the rack in darkness. In how many ways can the visitors take the hats such that nobody gets his own hat?

*Hints and answers.* 6. Pascal's triangle. 7. Find the coefficient of appropriate term in the product of this expression with its derivative. 8. Among elements  $1, 2, \dots, 2n+1$  consider the element with the largest number that belongs to the selection. 9. 24024. 10. 265.