

# MTAT.05.125 Introduction to Theoretical Computer Science

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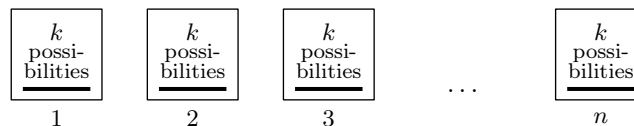
Lecture 2. Permutations and combinations with repetitions.  
Permutations of multisets.

## Permutations with repetitions

**Definition.** *Permutations with repetitions are ordered arrangements of  $k$  different elements in a row of length  $n$ , where the repetitions are allowed.*

Note: they are generally not permutations.

We have  $n$  positions:



For the first position there are  $k$  ways to put an element, for the second position there are also  $k$  ways to put an element, and so on. Using the multiplication principle, the total number of permutations with repetitions:

$$k \cdot k \cdot \dots \cdot k = k^n.$$

Example 1. There are  $26^5$  English words of length 5 (from last lecture).

## Permutations of multisets

We start with an example.

**Example 2.** How many different words (of length 11) can we construct from letters M, I, S, S, I, S, S, I, P, P, I?

*Solution.* We have the following letters:

| Letter | Amount |
|--------|--------|
| M      | 1      |
| I      | 4      |
| S      | 4      |
| P      | 2      |
| Total  | 11     |

Write all possible permutations, when assuming that all the letters are different. We have  $11!$  words. But now many words are in fact the same:

$$\begin{array}{c}
 M \underline{I}_1 S_1 S_2 \underline{I}_2 S_3 S_4 \underline{I}_3 P_1 P_2 \underline{I}_4 \\
 M \underline{I}_2 S_1 S_2 \underline{I}_1 S_3 S_4 \underline{I}_3 P_1 P_2 \underline{I}_4 \\
 \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\
 M \underline{I}_4 S_1 S_2 \underline{I}_3 S_3 S_4 \underline{I}_2 P_1 P_2 \underline{I}_1
 \end{array}$$

How many times did we write the same word:

- internal permutations of a letter I:  $4!$  possibilities.
- internal permutations of a letter S:  $4!$  possibilities.
- internal permutations of a letter P:  $2!$  possibilities.

Therefore, each word was written  $4!4!2!$  times. So

$$\text{number of words} \cdot 4!4!2! = 11!,$$

so

$$\text{number of words} = \frac{11!}{4!4!2!}.$$

More generally, if we have  $k_1$  objects of type 1,  $k_2$  objects of type 2,  $\dots$ ,  $k_t$  objects of type  $t$ , where  $k_1 + k_2 + \dots + k_t = n$ , then the number of possibilities to arrange them in the row is

$$\frac{n!}{k_1! k_2! \dots k_t!}.$$

**Example 3.** There are  $n$  students in a class. How many ways are there to divide them into teams of football, basketball and volleyball of  $k_1$ ,  $k_2$  and  $k_3$  students respectively? It is guaranteed that  $k_1 + k_2 + k_3 \leq n$ .

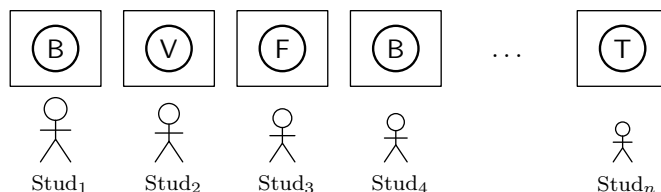
*Solution 1.* Valime järjest esimese, teise ja kolmanda võistkonna koosseisu.

- Choose  $k_1$  students for football team out of  $n$ , there are  $\binom{n}{k_1}$  possibilities.
- Choose  $k_2$  students for basketball team out of  $n - k_1$  left, there are  $\binom{n-k_1}{k_2}$  possibilities.
- Choose  $k_3$  students for volleyball team out of  $n - k_1 - k_2$  left, there are  $\binom{n-k_1-k_2}{k_3}$  possibilities.

Using rule of product, we have

$$\begin{aligned} \binom{n}{k_1} \binom{n-k_1}{k_2} \binom{n-k_1-k_2}{k_3} &= \frac{n!}{k_1!(n-k_1)!} \cdot \frac{(n-k_1)!}{k_2!(n-k_1-k_2)!} \cdot \frac{(n-k_1-k_2)!}{k_3!(n-k_1-k_2-k_3)!} \\ &= \frac{n!}{k_1!k_2!k_3!(n-k_1-k_2-k_3)!}. \end{aligned}$$

*Solution 2.* Arrange students in a row (for example according to their height). Take  $k_1$  football balls,  $k_2$  basketball balls,  $k_3$  volleyball balls and  $n - k_1 - k_2 - k_3$  tennis balls and arrange them in a row in front of the students:



There are

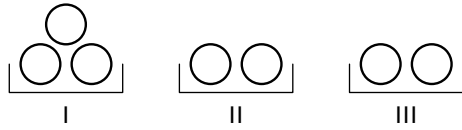
$$\frac{n!}{k_1!k_2!k_3!(n-k_1-k_2-k_3)!}$$

such arrangements, because those are permutations of multisets.

## Combinations with repetitions

**Example 4.** We have  $n$  boxes numbered by  $1, 2, \dots, n$ , and  $k$  identical balls. How many ways are there to distribute balls in the boxes.

For instance, if  $n = 3$  and  $k = 7$ , this is one example of such distribution:



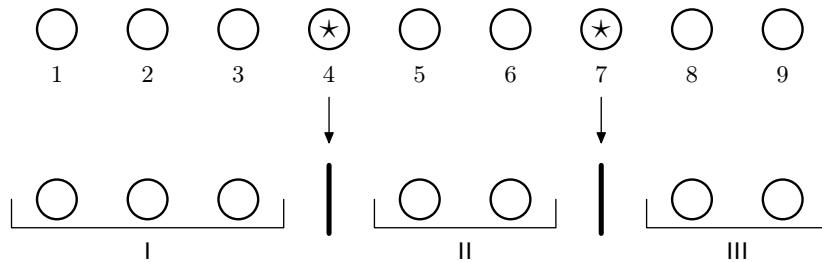
*Solution (Wrong!).* Put the balls in the row. Take  $n - 1$  partitions between the balls, as follows:



There are  $k - 1$  places to put partitions so we need to choose  $n - 1$  out of  $k - 1$  therefore the answer is  $\binom{k-1}{n-1}$ .

Wrong! Some boxes can be empty, i.e. two partitions at the same place but we don't take that into consideration.

*Solution (Correct).* Take  $n + k - 1$  balls and put them in the row. Now replace  $n - 1$  balls with partitions:



So we need to know how many ways exist to choose these  $n - 1$  balls out of  $n + k - 1$ . This is

$$\binom{n+k-1}{n-1} = \frac{(n+k-1)!}{(n-1)!k!} = \binom{n+k-1}{k}.$$

This is sometimes also denoted as  $CC_n^k$  or  $\left(\binom{n}{k}\right)$ .

**Summary** (combinations with permutations). We have  $n$  different types of elements. How many combinations of  $k$  elements exists, if we can choose the same element several times? Answer:  $\left(\binom{n}{k}\right)$ .

**Explanation.** Each ball “chooses” a box. There are  $k$  balls and  $n$  boxes. Balls are “identical” but the boxes are “different”. The order of choosing is not important, only the final result.

**Example 5.** How many integer solutions the following equation has:

$$x_1 + x_2 + \dots + x_n = k$$

when  $x_i \geq 0$  for all  $i = 1, 2, \dots, n$ ?

*Solution.* Each  $x_i$  is a “box”. We have  $k$  identical balls. If there are  $t_i$  balls in the box  $i$ , then we assign  $x_i = t_i$ . Answer:  $\binom{n}{k} = \binom{n+k-1}{k}$ .

## Summary

|  | without repetitions | with repetitions                  |
|--|---------------------|-----------------------------------|
| permutations                                     | $n!$                | $k^n$                             |
| $k$ -permutations /<br>permutations of multisets | $\frac{n!}{(n-k)!}$ | $\frac{n!}{k_1! k_2! \dots k_t!}$ |
| combinations                                     | $\binom{n}{k}$      | $\binom{n+k-1}{k}$                |

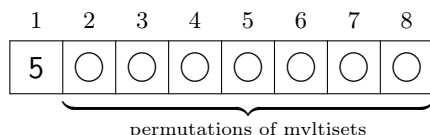
## Practice session

- How many 8-digit phone numbers exist that start with digit 5 and consist of digits 5, 5, 6, 6, 6, 7, 7, 7?
- How many different words of length 11 can we create from the letters M, I, S, S, I, S, S, I, P, P, I, where after M is always a letter I, and two letters P always appear together.
- There is a heap of red, blue, green and yellow balls. The balls of the same colour are identical. How many ways are there to choose 12 balls, such that there are *at least* 7 green balls?
- There is a heap of red, blue, green and yellow balls. The balls of the same colour are identical. How many ways are there to choose 12 balls, such that there are *at most* 7 green balls?
- Consider equation  $x_1 + x_2 + \dots + x_n = k$ , where  $x_1, x_2, \dots, x_n$  and  $k$  all integers. How many solutions does it have if for all  $i = 1, 2, \dots, n$  it should hold that  $x_i \in \{0, 1\}$ ?
- How many solutions exist in previous problem if for all  $i = 1, 2, \dots, n$  it should hold that  $x_i \geq 0$ ?
- How many solutions exist in previous problem if for all  $i = 1, 2, \dots, n$  it should hold that  $x_i \geq 1$ ?
- In some country there are 3 political parties running for parliament. In the election,  $2n + 1$  seats are distributed between these 3 parties. How many partitions of seats exist, such that any two parties together have a majority in the parliament?

## Solutions to the practice session questions

1. How many 8-digit phone numbers exist that start with digit 5 and consist of digits 5, 5, 6, 6, 6, 7, 7, 7?

*Solution.* Put 5 in the first position of the phone number. The remaining 7 digits can appear in an arbitrary order.



These are permutations of multisets. The answer is:

$$\frac{7!}{1! 3! 3!} = 140.$$

2. How many different words of length 11 can we create from the letters M, I, S, S, I, S, S, I, P, P, I, where after M is always a letter I, and two letters P always appear together.

*Solution.* We think about a combination of two letters M and I as one new letter MI, and a combination PP as a new letter too. Then we have to construct all words that consist of

- one letter MI,
- four letters S,
- three letters I,
- one letter PP.

Total length is 9. Using permutations of multisets, the answer is  $\frac{9!}{4!3!}$  (we ignore  $1! = 1$ ).

3. There is a heap of red, blue, green and yellow balls. The balls of the same colour are identical. How many ways are there to choose 12 balls, such that there are *at least* 7 green balls?

*Solution.* First, choose 7 green balls. Now, we have to choose the remaining 5 balls out of 4 types of balls. This can be done in

$$\binom{\binom{4}{5}}{\binom{5}{5}} = \binom{4+5-1}{5} = \binom{8}{5}$$

ways.

4. There is a heap of red, blue, green and yellow balls. The balls of the same colour are identical. How many ways are there to choose 12 balls, such that there are *at most* 7 green balls?

*Solution.*

- to choose 12 balls without restrictions:

$$\left(\binom{4}{12}\right) = \binom{4+12-1}{12} = \binom{15}{12}.$$

- to choose at least 8 green balls (similarly to the previous problem):

$$\left(\binom{4}{4}\right) = \binom{4+4-1}{4} = \binom{7}{4}$$

These are “forbidden” choices.

The answer is then the difference of these numbers:

$$\binom{15}{12} - \binom{7}{4}.$$

5. Consider equation  $x_1 + x_2 + \dots + x_n = k$ , where  $x_1, x_2, \dots, x_n$  and  $k$  all integers. How many solutions does it have if for all  $i = 1, 2, \dots, n$  it should hold that  $x_i \in \{0, 1\}$ ?

*Solution.* Choose those of  $x_1, x_2, \dots, x_n$  that are equal to 1 (it should be exactly  $k$  of them). There are  $\binom{n}{k}$  ways to do this.

6. How many solutions exist in previous problem if for all  $i = 1, 2, \dots, n$  it should hold that  $x_i \geq 0$ ?

*Solution.* The same as in lecture:  $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$ .

7. How many solutions exist in previous problem if for all  $i = 1, 2, \dots, n$  it should hold that  $x_i \geq 1$ ?

*Solution.* Again, we consider  $x_i$ 's as boxes and their values as number of balls we put in them. Then requirement  $x_i \geq 1$  means that we need to put at least one ball into each box. Do it and we will have  $k - n$  balls left to distribute between  $x_1, x_2, \dots, x_n$ . This can be done in

$$\left(\binom{n}{k-n}\right) = \binom{n+(k-n)-1}{k-n} = \binom{k-1}{k-n}$$

ways.

8. In some country there are 3 political parties running for parliament. In the election,  $2n + 1$  seats are distributed between these 3 parties. How many partitions of seats exist, such that any two parties together have a majority in the parliament?

*Solution.* The opposite from the condition that any two parties have a majority is the condition that there is a party which has at least  $n + 1$  seats.

- The number of different distributions of the seats to 3 parties is:

$$\left( \binom{3}{2n+1} \right) = \binom{3 + (2n+1) - 1}{2n+1} = \binom{2n+3}{2n+1}.$$

It is the same as to distribute  $2n + 1$  identical balls into 3 different boxes.

- The number of distributions where the first party has at least  $n + 1$  seats:

$$\left( \binom{3}{n} \right) = \binom{3 + n - 1}{n} = \binom{n+2}{n}.$$

The same amount is for the second and the third parties. These all are “forbidden” choices.

Then the result is:

$$\binom{2n+3}{2n+1} - 3 \binom{n+2}{n}.$$

## Additional exercises

9. How many 5-letter words can you form from letters A, B, C, D, E, if every letter should occur not more than two times?
10. How many ways are there to rearrange the letters of the word MATEMAATIKA, so that there are no letters A side by side? For example, this is correct: MATEMATAIKA.
11. How many ways are there to put 4 red and 6 blue balls into five different boxes?
12. There are 5 blue, 10 white and 15 red balls in the heap. How many ways are there to choose 10 balls from the heap (balls of the same colour are the same).

Answers in the same order: 2220, 88200, 14700, 51.