

MTAT.05.125 Introduction to Theoretical Computer Science

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Lecture 1. Permutations and combinations

Addition and multiplication principle

Solving combinatorial problems largely relies on two principles that can be used in many different situations. These are addition and multiplication principles.

Addition principle. *If we have n_a ways to do one thing and n_b ways to do another thing, then there are $n_a + n_b$ ways to do one of the two things.*

Example 1. Assume we can do the following things:

- playing dice with 6 sides – it gives 6 different outcomes (1, 2, 3, 4, 5, 6);
- flipping the coin with 2 sides – it gives 2 outcomes (heads, tails).

If we throw one of the objects – a dice or a coin – there are $6 + 2$ outcomes.

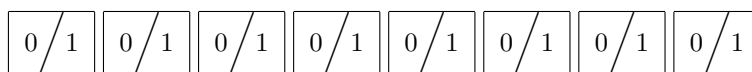
Multiplication principle. *If we can do one thing in n_a ways and other thing in n_b ways, then there are $n_a \cdot n_b$ ways to do both things.*

Example 2. If we throw a dice and a coin, then there are $6 \cdot 2 = 12$ outcomes:

(1, heads)	(1, tails)
(2, heads)	(2, tails)
(3, heads)	(3, tails)
(4, heads)	(4, tails)
(5, heads)	(5, tails)
(6, heads)	(6, tails)

Example 3. How many symbols can we represent by a binary vector of length 8?

Solution. Each of eight bits can have a value of either 0 or 1:

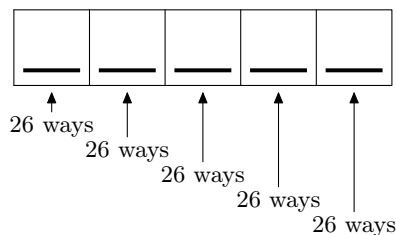


So in each position we could assign its value in two ways. Therefore by the multiplication principle the total number of ways to put zeroes and ones in 8 positions:

$$\underbrace{2 \cdot 2 \cdot \dots \cdot 2}_{8 \text{ times}} = 2^8 = 256.$$

Example 4. How many words of length 5 could be formed from letters A–Z?

Solution. There are 26 letters from A to Z. To form the word we need to fill in 5 positions:



According to multiplication principle, the total number of ways to form the word is:

$$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^5 = 11\,881\,376.$$

Example 5. How many 4-digit PIN codes exist that do not contain number 8?

Solution. We have 4 positions:

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. For each position we have 9 ways to put a digit (numbers 0–7 or 9). Then there are $9 \cdot 9 \cdot 9 \cdot 9 = 9^4 = 6561$ such different PIN codes.

Example 6. How many ways are there to choose 2 people of different nationalities from 5 French, 7 Germans and 3 Americans?

Solution. We can take

- French + German: $5 \cdot 7$ ways
- German + American: $7 \cdot 3$ ways
- French + American: $5 \cdot 3$ ways

Here we used multiplication principle three times. Further we use addition principle and get the final answer:

$$5 \cdot 7 + 7 \cdot 3 + 5 \cdot 3 = 35 + 21 + 15 = 71$$

ways.

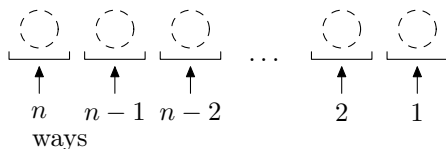
Permutations

Example 7. [*Permutations of n elements.*] Let there be n balls numbered 1 to n . How many ways to order them in a row?

For example if $n = 5$, this is an example of ordering the balls:



Solution.



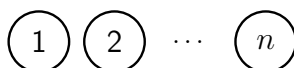
Ball for the first position we can choose in n different ways. After it has been fixed, we have $n - 1$ ways to choose the ball for the second position. Further, after balls for two first positions have been fixed, we have $n - 2$ ways to choose the ball for the third position. Continuing in this way, we get that the number of ways to order the balls in the row is

$$P(n) = n(n - 1)(n - 2) \dots \cdot 2 \cdot 1 = n!.$$

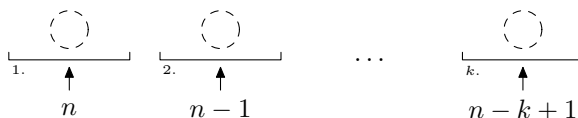
Definition. k -permutation is an ordered subset of size k of set of n elements.

Number of permutations is denoted as P_n^k or $P(n, k)$.

Let us count number of k -permutations. Assume we have n objects



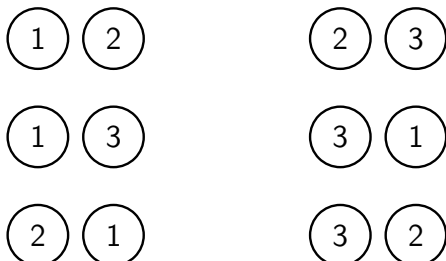
and k positions



Ball for the first position we can choose in n different ways. After it has been fixed, we have $n - 1$ ways to choose a ball for the second position. Further, after balls for two first positions have been fixed, we have $n - 2$ ways to choose a ball for the third position. Continuing in this way, we end up with $n - k + 1$ different ways to choose a ball for the last, k th, position. Therefore

$$P_n^k = n(n - 1) \dots (n - k + 1) = \frac{n!}{(n - k)!}.$$

Example 8. Assume we have 3 elements and want to count the number of ways we could form ordered subsets of size 2 (2-permutations) from these objects. Here $n = 3$ and $k = 2$ and, therefore, $P_3^2 = 6$. All 2-permutations are:



Combinations

Definition. k -combination is an unordered subset of size k of a set of size n .

Number of combinations is denoted as C_n^k or $C(n, k)$ or $\binom{n}{k}$.

Example 9. Assume we have 3 elements. These are all 2-combinations:

$$\left\{ \textcircled{1}, \textcircled{2} \right\} \quad \left\{ \textcircled{1}, \textcircled{3} \right\} \quad \left\{ \textcircled{2}, \textcircled{3} \right\}$$

And $\binom{3}{2} = 3$.

To choose an ordered subset of k elements from the set of n elements, we can act in two ways

1. Choose an ordered subset: P_n^k ways.
2. Choose an un-ordered subset (C_n^k ways) and then order it ($P(k)$ ways).
By the multiplication principle: $C_n^k \cdot P(k)$.

So it should hold: $P_n^k = C_n^k P(k)$. Then

$$C_n^k = \frac{P_n^k}{P(k)} = \frac{n!}{(n-k)!k!}.$$

Practice session

1. How many ways are there to arrange 10 students in a row?
2. How many ways are there to arrange 10 students in a circle?
3. The same question if Martin and Kadri should sit together.
4. The same question as 2, if it is forbidden that Janno and Robert sit together.
5. Project has 10 different tasks. We want to divide them between Karl and Kristjan. How many ways are there for doing so?
6. We want to divide a group of 6 tourists into two cars, red and blue. Each car takes at most 4 people. How many ways for doing so are there?
7. In the course Introduction to TCS there are $2n$ students. The teacher wants to divide them into pairs for the sake of doing homework assignments. How many ways for doing so are there?
8. How many ways are there to arrange n students in a row, such that Martin stands on the right side from Robert (see figure below)?
9. How many ways are there to arrange n students in a row, such that Martin is on the right of both Robert and Maria?
10. Prove that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

for $1 \leq k \leq n-1$.

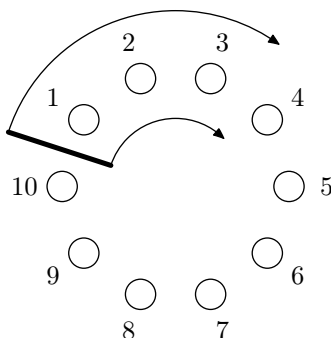
Solutions to practice session questions

1. How many ways are there to arrange 10 students in a row?

Solution. These are exactly permutations of 10 elements. Answer: $P(10) = 10!$ ways.

2. How many ways are there to arrange 10 students in a circle?

Solution. Denote this number by X . When the students are arranged in the circle, there are 10 ways to arrange them in the row (with the same relative order):



Then $10 \cdot X = 10!$ and $X = \frac{10!}{10} = 9!$.

3. The same question if Martin and Kadri should sit together.

Solution. Consider Martin and Kadri as “one object”. There are $8!$ ways to arrange now 9 objects in the circle. Now, there are two ways to arrange Martin-Kadri or Kadri-Martin. We obtain: $2 \cdot 8!$.

4. The same question as 2, if it is forbidden that Janno and Robert sit together.

Solution. There are two alternative cases: either two persons sit together, or not. Therefore

answer to problem 4 + answer to problem 3 = answer to problem 2,
which is the same as

answer to problem 4 = answer to problem 2 – answer to problem 3.

Answer: $9! - 2 \cdot 8! = 8!(9 - 2) = 7 \cdot 8!$.

5. Project has 10 different tasks. We want to divide them between Karl and Kristjan. How many ways are there for doing so?

Solution 1. If we know the tasks assigned to Karl, we automatically know the tasks assigned to Kristjan. For $k = 0, 1, 2, \dots, 10$ there are $\binom{10}{k}$ ways to assign Karl k tasks.

No of tasks assigned to Karl	No of ways to do this
0	$\binom{10}{0}$
1	$\binom{10}{1}$
2	$\binom{10}{2}$
\vdots	\vdots
10	$\binom{10}{10}$

In total there are $\binom{10}{0} + \binom{10}{1} + \dots + \binom{10}{10} = \sum_{k=0}^{10} \binom{10}{k}$ ways to assign tasks to Karl and it is exactly the number of ways to divide 10 tasks between Karl and Kristjan.

Solution 2. Every task could be assigned in two ways – either to Karl or to Kristjan.

Task 1	Task 2	Task 3	...	Task 10
Karl/	Karl/	Karl/	...	Karl/
Kristjan	Kristjan	Kristjan	...	Kristjan

According to multiplication principle, the total number of ways to do this is

$$2 \cdot 2 \cdot \dots \cdot 2 = 2^{10} = 1024.$$

If we solve the same problem for n tasks instead of 10, we get from Solution 1: $\sum_{i=0}^n \binom{n}{i}$. And from Solution 2: 2^n . This is the same number and therefore

$$\sum_{i=0}^n \binom{n}{i} = 2^n.$$

So by solving our problem from different perspectives we proved this well-known combinatorial identity.

6. We want to divide a group of 6 tourists into two cars, red and blue. Each car takes at most 4 people. How many ways for doing so are there?

Solution. Choose people for the red car: it should take between 2 and 4 people. Therefore,

$$\binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 15 + 20 + 15 = 50.$$

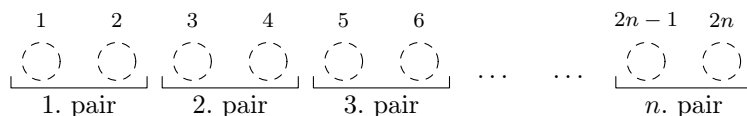
7. In the course Introduction to TCS there are $2n$ students. The teacher wants to divide them into pairs for the sake of doing homework assignments. How many ways for doing so are there?

Solution 1. At the beginning we have $2n$ students to choose from. So the first pair can be chosen in $\binom{2n}{2}$ ways. After that we have $(2n - 2)$ students left to choose from and the second pair can be chosen in $\binom{2n-2}{2}$ ways. And so on until the penultimate pair which can be chosen in $\binom{4}{2}$ ways.

However in this way we introduced the order between the different pairs. Each possibility was counted $n!$ times (number of permutations of n pairs when a pair is considered as a single object). Therefore the final solution is

$$\frac{\binom{2n}{2} \binom{2n-2}{2} \dots \binom{4}{2}}{n!}.$$

Solution 2. Order the students in a row:

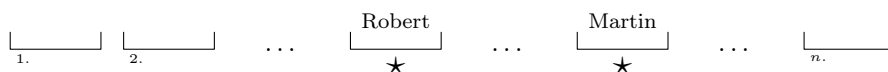


The first and the second form a pair, the third and the fourth form a pair and so on and so forth. Now, we need to cancel order between the pairs (like in previous solution) and within each pair. We obtain:

$$\frac{(2n)!}{n!(2!)^n}.$$

8. How many ways are there to arrange n students in a row, such that Martin stands on the right side from Robert (see figure below)?

Solution. Choose from n places 2 for Martin and Robert. There are $\binom{n}{2}$ ways to do that.



There are $(n-2)!$ ways to arrange the rest of the students. Using multiplication principle we get the answer: $\binom{n}{2}(n-2)!$.

9. How many ways are there to arrange n students in a row, such that Martin is on the right of both Robert and Maria?

Solution. First we choose 3 places for Martin, Maria and Robert without order – there are $\binom{n}{3}$ ways to do that. From 3 these places Martin should take the one on the right and there are two ways to place Maria and Robert on two other chosen seats. After they have their seats we have $(n-3)!$ way to place the rest students. We obtain:

$$\binom{n}{3} \cdot 2 \cdot (n-3)!$$

10. Prove that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

for $1 \leq k \leq n-1$.

Solution 1. Using definition of binomial coefficient, we get:

$$\begin{aligned} \binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{(n-k-1)!k!} \\ &= \frac{(n-1)!(k+n-k)}{(n-k)!k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}. \end{aligned}$$

Solution 2. Consider n balls numbered $1, 2, \dots, n$. Then $\binom{n}{k}$ is the number of ways to choose k of them without importance for order.

On the other hand, we can split the ways to choose k balls out of n into two cases¹:

- The ball number n is not chosen. Then in fact we are choosing k balls out of $n-1$. There are $\binom{n-1}{k}$ ways to do this.
- The ball number n is chosen. Then remaining $k-1$ balls should be chosen from $n-1$ balls left. There are $\binom{n-1}{k-1}$ ways to do that.

¹Depending on whether the ball number n was chosen or not.

Summing up these two cases is $\binom{n-1}{k-1} + \binom{n-1}{k}$, which is exactly the right hand side of the identity we are proving.

So we counted the number of ways to choose k balls out of n from two different perspectives. But since it is the same number, the values should be equal. This gives us the identity we are proving.

Additional exercises

11. Three students live in the same dormitory room. They have in all 4 cups, 5 plates, and 6 spoons (all different). In how many ways can they cover the dining table (each receives a cup, a plate, and a spoon).
12. 12 students participate in a course. The instructor has 8 different homework problems. In how many ways can the instructor split the students into groups of 3 and give each group a different homework problem?
13. In how many ways can 10 people be divided into 3 groups of as equal sizes as possible, i.e. each group has 3 or 4 people?
14. In how many ways can 4 men and 4 women be seated at a square table, such that each side has one man and one woman? Arrangements that can be obtained by rotating the table are considered the same.
15. In a four digit PIN code two digits are 4 and 8. How many PIN codes satisfy these conditions?

Answers to additional exercises in the same order: 172 800, 25 872 000, 2100, 2304, 672.