LTAT.04.001 Theoretical Computer Science

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Lecture 13. Classes P and NP.

The method of proof that we used for showing that HALT is undecidable is called "reduction from L_{TM}":

$$\begin{array}{ccc} \underbrace{L_{\text{TM}}} & \leq_{\text{M}} & \underbrace{\text{HALT}} \\ \text{can decide} & \Longleftarrow & \text{can decide} \end{array}$$

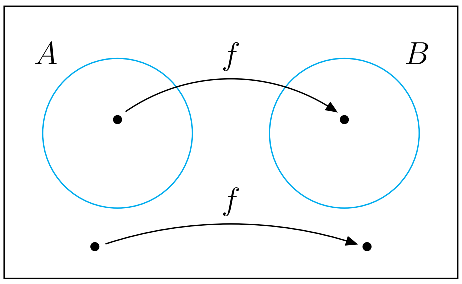
HALT is at least as hard as L_{TM}.

Definition. *Function $f\colon \Sigma^* \to \Sigma^*$ is a* computable function *if some Turing machine M on every input w halts with just $f(w)$ on its tape.*

Definition. *Language A is* mapping reducible *to language B, written $A \leq_M B$, if there is a computable function $f\colon \Sigma^* \to \Sigma^*$, where for every w*

$$w \in A \Leftrightarrow f(w) \in B.$$

The function f is called the reduction *from A to B.*



Theorem. *If $L_A \leq_M L_B$ and L_B is decidable, then L_A is decidable too.*

Proof. Let M_B be a Turing machine that decides L_B and f be a reduction from L_A to L_B. We describe a machine M_A that decides L_A:

1. On input w compute $f(w)$;
2. Run M_B on $f(w)$ and output what M_B outputs.

M_A decides language L_A indeed:

- If $w \in A$, then $f(w) \in B$, since f is reduction. M_B accepts $f(w)$ – therefore M_A accepts w.
- If $w \notin A$, then $f(w) \notin B$. M_B rejects $f(w)$ and therefore M_A rejects w. □

Polynomiality

Definition. *Let M be a deterministic Turing machine that halts on all inputs. The* running time (time complexity) *of M is the function $f\colon \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n.*

- $f(n)$ is the *running time* of M.
- Usually n is the length of the input.

Definition. *Let $f,\ g\colon \mathbb{N} \to \mathbb{R}^+$. We say that $f(n) = O(g(n))$, if ther are $C > 0$ and an integer n_0 such that for all $n > n_0$:*

$$f(n) < C\,g(n).$$

Example 1.

- If $f(n) = n + 10$, $g(n) = n^2$, then $f(n) = O(g(n))$, because if $n > 3$, then $n + 10 < n^2$. So we can take $C = 1$ and $n_0 = 3$.
- If $f(n) = 5n^3 + 2n^2 + 7n + 10$ and $g(n) = n^3$, then $f(n) = O(g(n))$. We can take $C = 6$ or $n_0 = 4$, or we can take $C = 100$ and $n_0 = 0$. As you see, the choice of C and n_0 is not unique.
- If $f(n) = 10n^2 + 100n + 10$ and $g(n) = 2^n$, then $f(n) = O(g(n))$.
- If $f(n) = \log_2 n$ and $g(n) = \sqrt{n}$, then $f(n) = O(g(n))$.

Since $\log_a n = \log_a b \cdot \log_b n$, then the basis of the logarithm is not important under O-notation (except exotic cases) and we simply write $O(\log n)$.

Definition. *Let $f,\ g\colon \mathbb{N} \to \mathbb{R}^+$. We say that $f(n) = o(g(n))$, if*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Example 2.

- $\sqrt{n} = o(n)$
- $\log n = o(n^{\frac{1}{100}})$
- $n \log n = o(n^2)$
- $n^{100} = o(2^n)$

Classes P and NP

For the following discussion, we do not distinguish between different polynomial complexities. However, we distinguish between polynomial and exponential time complexities. Different reasonable deterministic computational models are polynomially equivalent.

Definition. P *is a class of languages, whose time complexity is $f(n) = O(p(n))$, where p is some polynomial in n.*

P is invariant for all models of computation that are polynomially equivalent to the single-tape deterministic Turing machine. P roughly corresponds to the class of problems that are polynomially solvable on a computer.

Example 3. The following languages are in P.

- Any regular language.
- Given an array, find whether it is monotonically non-decreasing.
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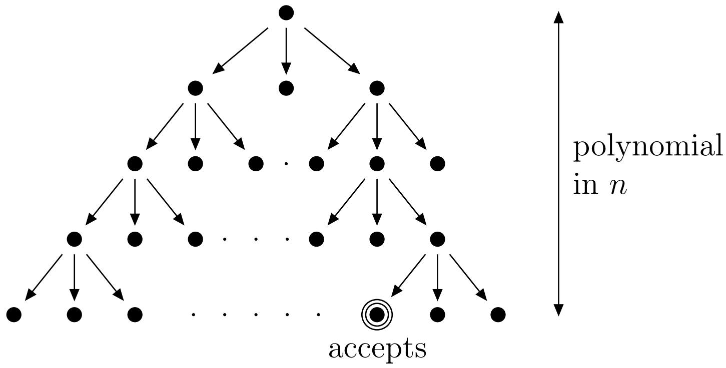
$$L_G = \{\langle G, s, t, k\rangle \mid \text{shortest path in graph } G \text{ from } s \text{ to } t \\ \text{has length } k\}.$$

Definition. NP *is the class of languages which are decided by some nondeterministic polynomial-time Turing machine.*



Example 4. A clique in a graph is a sub-graph, where every two nodes are connected by an edge. Then language

$$\text{CLIQUE} = \{\langle G, k\rangle \mid G \text{ is an undirected graph with clique of size } k\}.$$

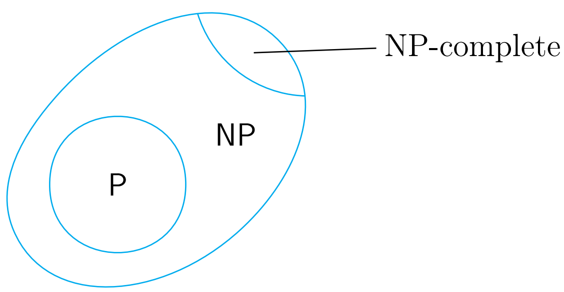
Let us show that CLIQUE $\in$ NP.

On input $\langle G, k\rangle$, the nondeterministic Turing machine does the following:

1. Nondeterministically selects a subset S of k nodes in G.
2. Checks whether G contains all the edges between pairs of nodes in S.
3. If yes – accepts, if not – rejects.

Trivially, $\mathsf{P} \subseteq \mathsf{NP}$. CLIQUE is an example of a problem, which is in NP, but it is not known if it is in P. It is know at all if $\mathsf{P} = \mathsf{NP}$, in other words it is not know if there exists problem which is in NP but not in P.

An NP-complete problem: if there exists a polynomial-time deterministic algorithm for that problem, then there exists a polynomial-time deterministic algorithm for any problem NP (i.e P=NP).



Practise session

1. Let

REGULAR = $\{\langle M \rangle \mid M$ is a Turing machine and $L(M)$ is a regular language$\}$

Prove that REGULAR is undecidable.
Solution. We show reduction:

$$L_{\text{TM}} \leq_{\text{M}} \text{REGULAR}.$$

Assume that REGULAR is decidable, and let M_R be a Turing machine that decides REGULAR. We construct Turing machine M_L that decides L_{TM}. On the input $\langle M, w \rangle$, M_L does the following:

1. Constructs machine M_0, which on input x does the following:
   (a) if x is of the form $0^n 1^n$ – accepts;
   (b) if x is not of the form $0^n 1^n$, run M on input w and accept if and only if M accepts.
2. Runs M_{R} on input $\langle M_0 \rangle$.
3. If M_{R} accepts – accept, if M_R rejects – reject.

What is the language of M_0?

- If M accepts w, then $L(M_0) = \Sigma^*$. This is regular language.
- If M does not accept w, then $L(M_0) = \{0^n 1^n \mid n \geq 0\}$. This is non-regular language.

Therefore:

- if M accepts w then $L(M_0) = \Sigma^*$ and M_R accepts $\langle M_0 \rangle$ in Step 2. Therefore M_L accepts.
- If M does not accept w then $L(M_0)$ is irregular and M_R rejects $\langle M_0 \rangle$ in Step 2. Therefore, M_L rejects.

So M_L accepts $\langle M, w \rangle$ if and only if M accepts w.

Note. All steps are computable by the Turing machines. In particular, constructing M_0 is possible: first M_0 checks for certain type of input and then simulates M on w.

Conclusion. We found that if there exists M_R (the machine that decides REGULAR), then there exists M_L (the machine that decides L_{TM}). Not possible. Contradiction!

2. True or false?

(a) $n = o(2n)$

(b) $3n^5 = O(10n^3 + 20n^2 + 100)$

(c) $2^n = o(3^n)$

(d) $n^2 = O(n \log n)$

Solution.

(a) False: $\lim\limits_{n\to\infty} \frac{n}{2n} = \frac{1}{2} \neq 0$.

(b) False: for any n_0 and $C > 0$ there exists arbitrary large $n \geq \max\{n_0 + 1, 1000C\}$, such that $n > n_0$ and $3n^5 > C(10n^3 + 20n^2 + 100)$.

(c) True: $\lim\limits_{n\to\infty} \frac{2^n}{3^n} = \lim\limits_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$.

(d) False: since $\lim\limits_{n\to\infty} \frac{n^2}{n \log n} = \lim\limits_{n\to\infty} \frac{n}{\log n} = \infty$, then for any constant $C > 0$ there exists n such that $f(n) > C\, g(n)$.

3. Let $t(n)$ be a function, where $t(n) \geq n$. Show that every $t(n)$-time multitape Turing machine has an equivalent $O(t^2(n))$-time single-tape machine.

Solution. We saw earlier in the course how to convert multitape machine into an equivalent single-tape machine. Now, we show that simulating each step of the multitape machine takes at most $O(t(n))$ steps of a single-tape machine.

Recall that initially the single-tape machine M_S puts on its tape the content of all tapes of the multi-tape machine M_M. To perform one step, M_S scans all the tape content to determine the symbols under the heads of M_M. Then, M_S makes another pass to write the new tape content. If one of the heads of M_M moves rightwards from the last non-blank symbol, M_S should shift the content of the tape one position to the right.

- What is the maximal number of steps for one scan? Since M_M makes $O(t(n))$ steps in total, the total length of the active part of the tape of M_S if of length $O(t(n))$. Hence each scan of the tape by M_S takes $O(t(n))$ time.

- To simulate each step of M_M, M_S performs two scans and possibly one shift to the right. Each such operation (scan/shift) takes at most $O(t(n))$.

- The total time needed for simulation of M_M by M_S:
  - initialisation of the tape: $O(t(n))$;
  - simulation of each of $t(n)$ steps of M_M by M_S: $t(n) \cdot O(t(n)) = O(t^2(n))$.

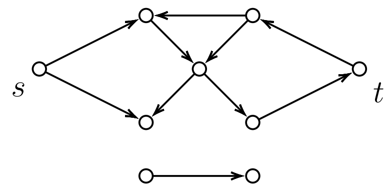
  Total time: $O(t^2(n)) + O(t(n))$.

  Since $O(t(n)) \geq O(n)$ (otherwise, M_M cannot even read all its input), we obtain that the total time complexity is $O(t^2(n))$.

4. Define the language:

$$\text{PATH} = \{\langle G, s, t\rangle \mid G \text{ is a directed graph} \\ \text{that has a directed path from } s \text{ to } t\}.$$

For example:



Prove that PATH $\in$ P.

Solution. Consider the following algorithm M for PATH, which on input $\langle G, s, t\rangle$ does the following:

1. Marks node s.
2. Repeats the following:
   - scans the edges (u, v) in graph G ; if u is marked and v is not marked – mark v.
3. If t is marked – accept, otherwise – reject.

Correctness. If there is a path $s = v_0 \to v_1 \to v_2 \to v_3 \to \ldots \to v_{l-1} \to v_l = t$, then by induction on i, the node v_i will be marked. If there is no path from s to t, the there is no way to mark t.

Complexity. Step 1 takes polynomial time. Let m be a number of nodes in G. Then Step 3 is executed at most m times (because each time we mark at least one node). Each of the steps 1–4 requires polynomial time complexity. Therefore, the total complexity is polynomial.