Final exam

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Student name: ______________________________

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1. This exam contains 10 pages. Check that no pages are missing.
2. It is possible to collect up to 110 points. Try to collect as many points as possible.
3. Justify and prove all your answers (where applicable).
4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.
5. Any printed and written material is allowed in the class. No electronic devices are allowed.
6. Exam duration is 2 hours.
7. Good luck!

Question 1	
Question 2	
Question 3	
Question 4	
Total	

Question 1 (30 points).

Define the language

$$\mathcal{L}_1 = \{\langle A \rangle \mid A \text{ is a DFA, and the language of } A \text{ is } (010)^n, n \geq 1\} \ .$$

Show that $\mathcal{L}_1$ is a decidable language.

Hint: one possible way to solve this question is to use the fact that $\mathcal{L}_\varnothing$ is a decidable language (shown in the lecture), where

$$\mathcal{L}_\varnothing = \{\langle A \rangle \mid A \text{ is a DFA and } L(A) = \varnothing\} \ .$$

Consider a DFA B that accepts all the strings except the strings of the form $(010)^n$, $n \geq 1$.

Question 2 (30 points).

Define the language

$$\mathcal{L}_2 = \{\langle \mathcal{M} \rangle \mid \mathcal{M} \text{ is a Turing machine, and the language of } \mathcal{M} \text{ is } (010)^n,\ n \geq 1\ \}\ .$$

In this question, you will show that $\mathcal{L}_2$ is an undecidable language.

<u>Hint:</u> for example, you can use a reduction from the language $\mathcal{L}_{\text{TM}}$. Assume that there exists a Turing machine $\mathcal{M}_2$ that decides $\mathcal{L}_2$. Construct a Turing machine $\mathcal{M}_{\text{TM}}$ that decides $\mathcal{L}_{\text{TM}}$, where

$$\mathcal{L}_{\text{TM}} = \{\langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a Turing machine and } \mathcal{M} \text{ accepts the input string } w\}\ .$$

On the input $\langle \mathcal{M}, w \rangle$, the machine $\mathcal{M}_{\text{TM}}$ does the following:

1. Constructs a machine $\mathcal{M}_w$, which on the input x does the following:
   (a) "simulates" the run of $\mathcal{M}$ on w;
   (b) if $\mathcal{M}$ rejects, – $\mathcal{M}_w$ rejects;
   (c) if $\mathcal{M}$ accepts, – $\mathcal{M}_w$ checks whether x has the form $(010)^n$, $n \geq 1$. If yes, – accepts. If not, – rejects.
2. Uses $\mathcal{M}_2$ to determine whether $L(\mathcal{M}_w) = \{(010)^n, n \geq 1\}$. If yes, – accepts. If not, – rejects.

Complete the details of the reduction if needed, and show that $\mathcal{L}_2$ is an undecidable language.

Question 3 (20 points).

Definition: a *frequent literal* in a CNF-formula ϕ is a literal that appears in at least half of all clauses.

Define a language $\mathsf{FREQUENT\text{-}LITERAL}$:

$$\mathsf{FREQUENT\text{-}LITERAL} = \{\langle \phi \rangle \mid \phi \text{ is a } \mathsf{CNF}\text{-formula that has a frequent literal}\} \ .$$

Is $\mathsf{FREQUENT\text{-}LITERAL} \in \mathcal{P}$? Justify your answer.

Question 4 (30 points).

Define a language 3-CLIQUES:

3-CLIQUES $= \{\langle \mathcal{G}, k \rangle \mid \mathcal{G}$ is an undirected graph that has at least 3 non-overlapping cliques of size $k\}$.

In this question, you will show that 3-CLIQUES is $\mathcal{NP}$-complete.

(a) Prove that 3-CLIQUES $\in \mathcal{NP}$.

(b) Prove that 3-CLIQUES is $\mathcal{NP}$-hard.

Hint: you can use a polynomial-time reduction from CLIQUE to 3-CLIQUES. Do not forget to show that the reduction is correct and polynomial-time.